Hidden Markov Models

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A Markov System

Has $N$ states, called $s_1$, $s_2$ .. $s_N$

There are discrete timesteps, $t=0$, $t=1$,

$N = 3$

t=0
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A Markov System

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There are discrete timesteps, \( t=0, t=1, \)

On the \( t \)'th timestep the system is in exactly one of the available states. Call it \( q_t \)

Note: \( q_t \in \{ s_1, s_2 .. s_N \} \)

\( N = 3 \)

\( t=0 \)

\( q_0 = s_3 \)

\( t=1 \)

\( q_1 = s_2 \)

Between each timestep, the next state is chosen randomly.
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The current state determines the probability distribution for the next state.

Often notated with arcs between states
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Markov Property

$q_{t+1}$ is conditionally independent of $\{q_{t-1}, q_{t-2}, \ldots, q_1, q_0\}$ given $q_t$.

In other words:

$$P(q_{t+1} = s_j | q_t = s_i) = P(q_{t+1} = s_j | q_t = s_i, \text{any earlier history})$$

Question: what would be the best Bayes Net structure to represent the Joint Distribution of $(q_0, q_1, \ldots, q_3, q_4)$?
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$q_{t+1}$ is conditionally independent of \{ $q_{t-1}$, $q_{t-2}$, ..., $q_1$, $q_0$ \} given $q_t$.

In other words:

$P(q_{t+1} = s_j | q_t = s_i) = \frac{1}{3}$

$P(q_{t+1} = s_j | q_t = s_i, \text{any earlier history})$

Question: what would be the best Bayes Net structure to represent the Joint Distribution of ($q_0$, $q_1$, $q_2$, $q_3$, $q_4$)?

Answer:
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In other words:

$$P(q_{t+1} = s_j | q_t = s_i, \text{any earlier history}) = P(q_{t+1} = s_j | q_t = s_i)$$

**Question:** What would be the best Bayes Net structure to represent the Joint Distribution of $(q_0, q_1, q_2, q_3, q_4)$?

**Answer:**

Each of these probability tables is identical:

$$P(q_{t+1} = s_j | q_t = s_i)$$

Notation:

$$a_{ij} = P(q_{t+1} = s_j | q_t = s_i)$$

### A Blind Robot

A human and a robot wander around randomly on a grid…

**STATE $q =$** Location of Robot, Location of Human

**Note:** $N$ (num. states) = 18 \(\times\) 18 = 324
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A Blind Robot

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Dynamics of System

$q_0 = \begin{array}{c|c|c}
 & R & \\
 H & & \\
 & & \\
\end{array}$

Each timestep the human moves randomly to an adjacent cell. And Robot also moves randomly to an adjacent cell.

Typical Questions:

• “What’s the expected time until the human is crushed like a bug?”
• “What’s the probability that the robot will hit the left wall before it hits the human?”
• “What’s the probability Robot crushes human on next time step?”

Example Question

“It’s currently time $t$, and human remains uncrushed. What’s the probability of crushing occurring at time $t + 1$?”

If robot is blind:
We can compute this in advance.

If robot is omnipotent:
(I.E. If robot knows state at time $t$), can compute directly.

If robot has some sensors, but incomplete state information …
Hidden Markov Models are applicable!

We’ll do this first
Too Easy. We won’t do this
Main Body of Lecture
Dynamics of System

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Main Body of Lecture
What is \( P(q_t = s) \)? slow, stupid answer

Step 1: Work out how to compute \( P(Q) \) for any path \( Q = q_1 q_2 q_3 \ldots q_t \)

Given we know the start state \( q_1 \) (i.e. \( P(q_1) = 1 \))

\[
P(q_1 q_2 \ldots q_t) = P(q_1 q_2 \ldots q_{t-1}) P(q_t|q_1 q_2 \ldots q_{t-1})
\]

\[
= P(q_1 q_2 \ldots q_{t-1}) P(q_t|q_{t-1})
\]

\[
= P(q_2|q_1)P(q_3|q_2)\ldots P(q_t|q_{t-1})
\]

Step 2: Use this knowledge to get \( P(q_t = s) \)

\[
P(q_t = s) = \sum_{Q \text{ Paths of length } t \text{ that end in } s} P(Q)
\]

What is \( P(q_t = s) \)? Clever answer

- For each state \( s_i \), define
  \[
p_t(i) = \text{Prob. state is } s_i \text{ at time } t
  = P(q_t = s_i)
\]
- Easy to do inductive definition
  \[
\forall i \quad p_0(i) =
\]
  \[
\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =
\]
What is $P(q_t = s)$? slow, stupid answer

Step 1: Work out how to compute $P(Q)$ for any path $Q$

$Q = q_1 \ q_2 \ q_3 .. q_t$

Given we know the start state $q_1$ (i.e. $P(q_1) = 1$)

$P(q_1 \ q_2 .. q_t) = P(q_1 \ q_2 .. q_{t-1}) \ P(q_t | q_{t-1})$

$= P(q_2 | q_1) \ P(q_3 | q_2) ... P(q_t | q_{t-1})$

WHY?

Step 2: Use this knowledge to get $P(q_t = s)$

$P(q_t = s) = \sum_{Q:\text{Paths of length } t \text{ that end in } s} P(Q)$

Computation is exponential in $t$

What is $P(q_t = s)$? Clever answer

• For each state $s_i$, define

$p_t(i) = \text{Prob. state is } s_i \text{ at time } t$

$= P(q_t = s_i)$

• Easy to do inductive definition

$\forall i \ p_0(i) =$

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What is $P(q_t=s)$? Clever answer

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  $\forall i \ p_0(i) = \begin{cases} 1 & \text{if } s_i \text{ is the start state} \\ 0 & \text{otherwise} \end{cases}$

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  $$\sum_{i=1}^{N} P(q_{t+1} = s_j \land q_t = s_i) =$$

  $$\sum_{i=1}^{N} P(q_{t+1} = s_j \mid q_t = s_i) P(q_t = s_i) = \sum_{i=1}^{N} a_{ij} p_t(i)$$

**Remember,**

$$a_{ij} = P(q_{t+1} = s_j \mid q_t = s_i)$$

---

<table>
<thead>
<tr>
<th>$t$</th>
<th>$p_t(1)$</th>
<th>$p_t(2)$</th>
<th>...</th>
<th>$p_t(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
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<td>$t_{final}$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

- Computation is simple.
- Just fill in this table in this order:
What is $P(q_t = s)$? Clever answer

- For each state $s_j$, define
  
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  • Cost of computing \( p_t(i) \) for all states \( S_i \) is now \( O(t N^2) \)
  • The stupid way was \( O(N^t) \)
  • This was a simple example
  • It was meant to warm you up to this trick, called Dynamic Programming, because HMMs do many tricks like this.

---

Hidden State

“It’s currently time \( t \), and human remains uncrushed. What’s the probability of crushing occurring at time \( t + 1 \)?”

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- If robot has some sensors, but incomplete state information …
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Main Body of Lecture
What is $P(q_t = s)$ ? Clever answer

- For each state $s_i$, define
  
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Main Body of Lecture
Hidden State

• The previous example tried to estimate $P(q_t = s_i)$ unconditionally (using no observed evidence).

• Suppose we can observe something that’s affected by the true state.

• Example: **Proximity sensors.** (tell us the contents of the 8 adjacent squares)

Noisy Hidden State

• Example: **Noisy Proximity sensors.** (unreliably tell us the contents of the 8 adjacent squares)
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True state $q_t$

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W denotes “WALL”

What the robot sees: Observation $O_t$
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Noisy Hidden State

- Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)

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True state \( q_t \)

\( O_t \) is noisily determined depending on the current state.

Assume that \( O_t \) is conditionally independent of \( \{ q_{t-1}, q_{t-2}, \ldots q_1, q_0, O_{t-1}, O_{t-2}, \ldots O_1, O_0 \} \) given \( q_t \).

In other words:

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P(O_t = X | q_t = s_i) = \\
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**Uncorrupted Observation**

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What the robot sees: Observation $O_t$

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Noisy Hidden State

Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)

- True state: $q_t$
- Uncorrupted Observation: $H$
- What the robot sees: Observation $O_t$

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Answer:
Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)

- **E x a m p l e:** Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)
  - Uncorrupted Observation
  - What the robot sees: Observation $O_t$
  - $O_t$ is noisily determined depending on the current state.
  - Assume that $O_t$ is conditionally independent of $\{q_{t-1}, q_{t-2}, \ldots, q_1, q_0, O_{t-1}, O_{t-2}, \ldots, O_1, O_0\}$ given $q_t$.
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Question: what’d be the best Bayes Net structure to represent the Joint Distribution of $(q_0, q_1, q_2, q_3, q_4, O_0, O_1, O_2, O_3, O_4)$?

**Answer:**

- Notation: $b_i(k) = P(O_t = k | q_t = s_i)$

- Table:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$b_1(1)$</th>
<th>$b_1(2)$</th>
<th>$b_1(k)$</th>
<th>$b_1(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p(O=1</td>
<td>q_0=s_0)$</td>
<td>$p(O=2</td>
<td>q_0=s_0)$</td>
</tr>
<tr>
<td>2</td>
<td>$b_2(1)$</td>
<td>$b_2(2)$</td>
<td>$b_2(k)$</td>
<td>$b_2(M)$</td>
</tr>
<tr>
<td>3</td>
<td>$b_3(1)$</td>
<td>$b_3(2)$</td>
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<td>$b_3(M)$</td>
</tr>
<tr>
<td>4</td>
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</tr>
</tbody>
</table>

- Question: what’d be the best Bayes Net structure to represent the Joint Distribution of $(q_0, q_1, q_2, q_3, q_4, O_0, O_1, O_2, O_3, O_4)$?
Hidden Markov Models

Our robot with noisy sensors is a good example of an HMM

- **Question 1: State Estimation**
  What is \( P(q_T=S_i | O_1O_2...O_T) \)?
  It will turn out that a new cute D.P. trick will get this for us.

- **Question 2: Most Probable Path**
  Given \( O_1O_2...O_T \), what is the most probable path that I took?
  And what is that probability?
  Yet another famous D.P. trick, the VITERBI algorithm, gets this.

- **Question 3: Learning HMMs:**
  Given \( O_1O_2...O_T \), what is the maximum likelihood HMM that could have produced this string of observations?
  Very very useful. Uses the E.M. Algorithm

Are H.M.M.s Useful?

You bet !!

- Robot planning + sensing when there’s uncertainty (e.g. Reid Simmons / Sebastian Thrun / Sven Koenig)
- Speech Recognition/Understanding
  Phones \( \rightarrow \) Words, Signal \( \rightarrow \) phones
- Human Genome Project
  Complicated stuff your lecturer knows nothing about.
- Consumer decision modeling
- Economics & Finance.
  Plus at least 5 other things I haven’t thought of.
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Some Famous HMM Tasks

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Woke up at 8.35, Got on Bus at 9.46, Sat in lecture 10.05-11.22…
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## Basic Operations in HMMs

For an observation sequence \( O = O_1 \ldots O_T \), the three basic HMM operations are:

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<th>Problem</th>
<th>Algorithm</th>
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<tr>
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<td>( O(TN^2) )</td>
</tr>
<tr>
<td><strong>Inference:</strong> Computing ( Q^* = \arg\max_Q P(Q \mid O) )</td>
<td>Viterbi Decoding</td>
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<td><strong>Learning:</strong> Computing ( \lambda^* = \arg\max_{\lambda} P(O \mid \lambda) )</td>
<td>Baum-Welch (EM)</td>
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\( T = \# \text{ timesteps}, N = \# \text{ states} \)
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\( T = \# \) timesteps, \( N = \# \) states
HMM Notation  
(from Rabiner’s Survey)

The states are labeled $S_1, S_2, \ldots, S_N$

For a particular trial….  

Let $T$ be the number of observations  

$T$ is also the number of states passed through  

$O = O_1 O_2 \ldots O_T$ is the sequence of observations  

$Q = q_1 q_2 \ldots q_T$ is the notation for a path of states  

$\lambda = \langle N, M, \{\pi_i\}, \{a_{ij}\}, \{b_i(j)\}\rangle$ is the specification of an HMM

---

HMM Formal Definition

An HMM, $\lambda$, is a 5-tuple consisting of  

- $N$ the number of states  
- $M$ the number of possible observations  
- $\{\pi_1, \pi_2, \ldots, \pi_N\}$ The starting state probabilities  
  $P(q_0 = S_i) = \pi_i$  
- $\{a_{ij}\}$ The state transition probabilities  
  $P(q_{t+1} = S_j \mid q_t = S_i) = a_{ij}$  
- $\{b_i(k)\}$ The observation probabilities  
  $P(O_t = k \mid q_t = S_i) = b_i(k)$
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  $P(q_0 = S_i) = \pi_i$

- $a_{11} a_{22} ... a_{1N}$
  $a_{21} a_{22} ... a_{2N}$
  $: : :$
  $a_{N1} a_{N2} ... a_{NN}$
  The state transition probabilities
  $P(q_{t+1} = S_j | q_t = S_i) = a_{ij}$

- $b_1(1) b_1(2) ... b_1(M)$
  $b_2(1) b_2(2) ... b_2(M)$
  $: : :$
  $b_N(1) b_N(2) ... b_N(M)$
  The observation probabilities
  $P(O_t = k | q_t = S_i) = b_i(k)$
Here’s an HMM

N = 3
M = 3
π₁ = ½
π₂ = ½
π₃ = 0

a₁₁ = 0
a₁₂ = ⅓
a₁₃ = ⅔
da₂₁ = ⅓
da₂₂ = 0
da₂₃ = ⅓
da₃₁ = ⅓
da₃₂ = ⅓
da₃₃ = ⅓

b₁ (X) = ½
b₁ (Y) = ½
b₁ (Z) = 0
b₂ (X) = 0
b₂ (Y) = ½
b₂ (Z) = ½
b₃ (X) = ½
b₃ (Y) = 0
b₃ (Z) = ½

Let’s generate a sequence of observations:

50-50 choice between S₁ and S₂

q₀ = ___ O₀ = ___
q₁ = ___ O₁ = ___
q₂ = ___ O₂ = ___
Here’s an HMM

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.

N = 3
M = 3
π₁ = ½
π₂ = ½
π₃ = 0

a₁₁ = 0
a₁₂ = ½
a₁₃ = ½

a₁₂ = ½
a₂₂ = 0
a₁₃ = ½

a₁₃ = ½
a₃₂ = ½
a₁₃ = ½

b₁ (X) = ½
b₁ (Y) = ½
b₁ (Z) = 0

b₂ (X) = ½
b₂ (Y) = ½
b₂ (Z) = ½

b₃ (X) = ½
b₃ (Y) = ½
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<table>
<thead>
<tr>
<th>q₀</th>
<th>O₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>q₁</td>
<td>O₁</td>
</tr>
<tr>
<td>q₂</td>
<td>O₂</td>
</tr>
</tbody>
</table>
Here's an HMM

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

\[ q_0 = S_1 \quad O_0 = \_
\]
\[ q_1 = \_ \quad O_1 = \_
\]
\[ q_2 = \_ \quad O_2 = \_
\]

N = 3
M = 3
\( \pi_1 = \frac{1}{2} \quad \pi_2 = \frac{1}{2} \quad \pi_3 = 0 \)
\[ a_{11} = 0 \quad a_{12} = \frac{1}{3} \quad a_{13} = \frac{2}{3} \]
\[ a_{12} = \frac{1}{3} \quad a_{22} = 0 \quad a_{13} = \frac{1}{3} \]
\[ a_{13} = \frac{2}{3} \quad a_{32} = \frac{1}{3} \quad a_{13} = \frac{1}{3} \]
\[ b_1 (X) = \frac{1}{2} \quad b_1 (Y) = \frac{1}{2} \quad b_1 (Z) = 0 \]
\[ b_2 (X) = 0 \quad b_2 (Y) = \frac{1}{2} \quad b_2 (Z) = \frac{1}{2} \]
\[ b_3 (X) = \frac{1}{2} \quad b_3 (Y) = 0 \quad b_3 (Z) = \frac{1}{2} \]
Here’s an HMM

N = 3  
M = 3  
π₁ = ½  
π₂ = ½  
π₃ = 0  
a₁₁ = 0  
a₁₂ = ½  
a₁₃ = ½  
a₂₁ = ½  
a₂₂ = 0  
a₂₃ = ½  
a₃₁ = ½  
a₃₂ = ½  
a₃₃ = 0  
b₁ (X) = ½  
b₁ (Y) = ½  
b₁ (Z) = 0  
b₂ (X) = 0  
b₂ (Y) = ½  
b₂ (Z) = ½  
b₃ (X) = ½  
b₃ (Y) = ½  
b₃ (Z) = ½

Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let’s generate a sequence of observations:

<table>
<thead>
<tr>
<th>( q_0 )</th>
<th>( S_1 )</th>
<th>( O_0 )</th>
<th>( _ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>_</td>
<td>( O_1 )</td>
<td>_</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>_</td>
<td>( O_2 )</td>
<td>_</td>
</tr>
</tbody>
</table>

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Here’s an HMM

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.
Let’s generate a sequence of observations:

N = 3
M = 3
π₁ = \(\frac{1}{2}\)  π₂ = \(\frac{1}{2}\)  π₃ = 0

\[\begin{align*}
a_{11} &= 0 & a_{12} &= \frac{1}{3} & a_{13} &= \frac{2}{3} \\
a_{12} &= \frac{1}{3} & a_{22} &= 0 & a_{13} &= \frac{2}{3} \\
a_{13} &= \frac{1}{3} & a_{32} &= \frac{1}{3} & a_{13} &= \frac{1}{3} \\
b_{1}(X) &= \frac{1}{2} & b_{1}(Y) &= \frac{1}{2} & b_{1}(Z) &= 0 \\
b_{2}(X) &= 0 & b_{2}(Y) &= \frac{1}{2} & b_{2}(Z) &= \frac{1}{2} \\
b_{3}(X) &= \frac{1}{2} & b_{3}(Y) &= 0 & b_{3}(Z) &= \frac{1}{2} \\
\end{align*}\]
Here’s an HMM

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.

Let’s generate a sequence of observations:

<table>
<thead>
<tr>
<th>State 0</th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>q_0=</td>
<td>S_1</td>
<td>O_0= X</td>
</tr>
<tr>
<td>q_1=</td>
<td>S_3</td>
<td>O_2=</td>
</tr>
<tr>
<td>q_2=</td>
<td>_</td>
<td>O_2= _</td>
</tr>
</tbody>
</table>

Each of the three next states is equally likely

N = 3
M = 3
\( \pi_1 = \frac{1}{2} \) \( \pi_2 = \frac{1}{2} \) \( \pi_3 = 0 \)

\[ a_{11} = 0 \quad a_{12} = \frac{1}{3} \quad a_{13} = \frac{2}{3} \]
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Here’s an HMM

N = 3
M = 3
π₁ = ½
π₂ = ½
π₃ = 0

a₁₁ = 0
a₁₂ = ⅓
a₁₃ = ⅔
a₁₂ = ⅓
a₂₂ = 0
a₁₃ = ⅓
a₁₃ = ⅓
a₃₂ = ⅓
a₁₃ = ⅓

b₁ (X) = ½
b₁ (Y) = ⅔
b₁ (Z) = 0
b₁ (X) = ⅓
b₁ (Y) = ⅔
b₁ (Z) = 0
b₁ (X) = ⅓
b₁ (Y) = ⅔
b₁ (Z) = 0

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.

Let’s generate a sequence of observations:

50-50 choice between Z and X

S₁
S₂
S₃

O₀ = X
O₀ = X
O₀ = Z

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Here's an HMM

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.
Let's generate a sequence of observations:

N = 3
M = 3
π₁ = ½ π₂ = ½ π₃ = 0

a₁₁ = 0 a₁₂ = ⅓ a₁₃ = ⅔
da₁₂ = ⅓ a₂₂ = 0 a₁₃ = ⅔
da₁₃ = ⅓ a₃₂ = ½ a₁₃ = ⅔
da₁₃ = ⅓ a₃₂ = ½ a₁₃ = ⅔
db₁ (X) = ½ db₁ (Y) = ½ db₁ (Z) = 0
db₂ (X) = 0 db₂ (Y) = ½ db₂ (Z) = ½
db₃ (X) = ½ db₃ (Y) = 0 db₃ (Z) = ½

50-50 choice between Z and X

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State Estimation

Start randomly in state 1 or 2
Choose one of the output symbols in each state at random.
Let's generate a sequence of observations:

This is what the observer has to work with...

N = 3
M = 3
π₁ = ½
π₂ = ½
π₃ = 0

a₁₁ = 0
a₁₂ = ½
a₁₃ = ½

a₂₁ = ½
a₂₂ = 0
a₂₃ = ½

a₃₁ = ½
a₃₂ = ½
a₃₃ = ½

b₁₁ (X) = ½
b₁₂ (Y) = ½
b₁₃ (Z) = 0

b₂₁ (X) = 0
b₂₂ (Y) = ½
b₂₃ (Z) = ½

b₃₁ (X) = ½
b₃₂ (Y) = 0
b₃₃ (Z) = ½

Prob. of a series of observations

What is \( P(O) = P(O_1 O_2 O_3) = P(O_1 = X \land O_2 = X \land O_3 = Z) \)?

Slow, stupid way:

\[
P(O) = \sum_{Q: \text{Paths of length } 3} P(O \land Q)
= \sum_{Q: \text{Paths of length } 3} P(O|Q)P(Q)
\]

How do we compute \( P(Q) \) for an arbitrary path \( Q \)?

How do we compute \( P(O|Q) \) for an arbitrary path \( Q \)?
State Estimation

Start randomly in state 1 or 2
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Let’s generate a sequence of observations:

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<td>X</td>
<td>?</td>
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Prob. of a series of observations

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P(O) = \sum_{Q:\text{Paths of length }3} P(O \land Q)
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How do we compute \( P(O \mid Q) \) for an arbitrary path \( Q \)?

Example in the case \( Q = S_1 S_3 S_3 \):
\[
= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}
\]

Prob. of a series of observations

What is \( P(O) = P(O_1 O_2 O_3) = P(O_1 = X \land O_2 = X \land O_3 = Z) \)?

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P(O) = \sum_{Q:\text{Paths of length }3} P(O \land Q)
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= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}
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Prob. of a series of observations

What is $P(O) = P(O_1 \cdot O_2 \cdot O_3)$ = $P(O_1 = X \quad O_2 = X \quad O_3 = Z)$?

Slow, stupid way:

$$P(O) = \sum_{Q \text{ Paths of length 3}} P(O \land Q)$$

$$= \sum_{Q \text{ Paths of length 3}} P(O | Q)P(Q)$$

How do we compute $P(Q)$ for an arbitrary path $Q$?

How do we compute $P(O|Q)$ for an arbitrary path $Q$?

Example in the case $Q = S_1 S_3 S_3$:

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{9}$$

---

Prob. of a series of observations

What is $P(O) = P(O_1 \cdot O_2 \cdot O_3)$ = $P(O_1 = X \quad O_2 = X \quad O_3 = Z)$?

Slow, stupid way:

$$P(O) = \sum_{Q \text{ Paths of length 3}} P(O \land Q)$$

$$= \sum_{Q \text{ Paths of length 3}} P(O | Q)P(Q)$$

How do we compute $P(Q)$ for an arbitrary path $Q$?

How do we compute $P(O|Q)$ for an arbitrary path $Q$?

Example in the case $Q = S_1 S_3 S_3$:

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$
The Prob. of a given series of observations, non-exponential-cost-style

Given observations $O_1, O_2 \ldots O_T$

Define

$$\alpha_t(i) = P(O_1, O_2 \ldots O_t \land q_t = S_i \mid \lambda) \quad \text{where } 1 \leq t \leq T$$

$\alpha_t(i) =$ Probability that, in a random trial,

- We’d have seen the first $t$ observations
- We’d have ended up in $S_i$ as the $t$’th state visited.

In our example, what is $\alpha_2(3)$ ?
Prob. of a series of observations

What is $P(O) = P(O_1, O_2, O_3) = P(O_1 = X, O_2 = X, O_3 = Z)$?

Slow, stupid way:

$$P(O) = \sum_{Q: \text{Paths of length 3}} P(O \wedge Q)$$

$$= \sum_{Q: \text{Paths of length 3}} P(O | Q) P(Q)$$

How do we compute $P(Q)$ for an arbitrary path $Q$?

How do we compute $P(O | Q)$ for an arbitrary path $Q$?

The Prob. of a given series of observations, non-exponential-cost-style

Given observations $O_1, O_2, \ldots, O_T$

Define

$$\alpha_t(i) = P(O_1, O_2, \ldots, O_t \wedge q_t = S_i | \lambda) \quad \text{where } 1 \leq t \leq T$$

$\alpha_t(i)$ = Probability that, in a random trial,

- We’d have seen the first $t$ observations
- We’d have ended up in $S_i$ as the $t$'th state visited.

In our example, what is $\alpha_2(3)$?
\( \alpha_t(i) \): easy to define recursively

\[ \alpha_t(i) = \Pr(O_1, O_2, \ldots, O_T \land q_i = S_i \mid \lambda) \] (\( \alpha_t(i) \) can be defined stupidly by considering all paths length \( t \). How?)

\[ \alpha_t(i) = \Pr(O_i \land q_i = S_i) \]
\[ = \Pr(q_i = S_i) \Pr(O_i \mid q_i = S_i) \]
\[ = \text{what?} \]
\[ \alpha_{t+1}(j) = \Pr(O_{t+1} \ldots O_T \land q_{t+1} = S_j) \]
\[ = \sum_{i=1}^{N} \Pr(O_{t+1} \ldots O_T \land q_{t+1} = S_j \land O_i \land q_i = S_i) \]
\[ = \sum_{i=1}^{N} \Pr(O_{t+1} \land q_{t+1} = S_j \mid q_i = S_i) \alpha_t(i) \]
\[ = \sum_{i=1}^{N} \Pr(q_{t+1} = S_j \mid q_i = S_i) \Pr(O_{t+1} \mid q_{t+1} = S_j) \alpha_t(i) \]
\[ = \sum_{i=1}^{N} a_j b_j(O_{t+1}) \alpha_t(i) \]
\( \alpha_t(i) \): easy to define recursively

\[ \alpha_t(i) = P(O_1 O_2 \ldots O_T \land q_t = S_i | \lambda) \]

\( \alpha_t(i) \) can be defined stupidly by considering all paths length \( t \). How?

\[
\begin{align*}
\alpha_t(i) & = P(O_t \land q_t = S_i) \\
& = P(q_t = S_i)P(O_t | q_t = S_i) \\
& = \text{what?}
\end{align*}
\]

\[
\begin{align*}
\alpha_{t+1}(j) & = P(O_1 O_2 \ldots O_{t+1} \land q_{t+1} = S_j) \\
& = \sum_{i=1}^{N} P(O_1 O_2 \ldots O_t \land q_i = S_i \land O_{t+1} \land q_{t+1} = S_j) \\
& = \sum_{j=1}^{N} P(O_{t+1}, q_{t+1} = S_j | O_1 O_2 \ldots O_t \land q_t = S_i)P(O_t \ldots O_{t+1} \land q_{t+1} = S_j) \\
& = \sum_{i} P(O_{t+1}, q_{t+1} = S_j | q_i = S_i)\alpha_t(i) \\
& = \sum_{i} a_{j|i} b_j(O_{t+1}) \alpha_t(i)
\end{align*}
\]
in our example

\[
\alpha_i(i) = P(O_1O_2\ldots O_t \land q_t = S_i | \lambda)
\]
\[
\alpha_i(i) = b_i(O_1)\pi_i
\]
\[
\alpha_{t+1}(j) = \sum_i a_{ij} b_j(O_{t+1})\alpha_i(i)
\]

We saw \( O_1 O_2 O_3 = X X Z \)

\[
\begin{align*}
\alpha_1(1) &= \frac{1}{4} & \alpha_1(2) &= 0 & \alpha_1(3) &= 0 \\
\alpha_2(1) &= 0 & \alpha_2(2) &= 0 & \alpha_2(3) &= \frac{1}{12} \\
\alpha_3(1) &= 0 & \alpha_3(2) &= \frac{1}{72} & \alpha_3(3) &= \frac{1}{72}
\end{align*}
\]

Easy Question

We can cheaply compute

\[
\alpha_t(i) = P(O_1O_2\ldots O_t \land q_t = S_i)
\]

(How) can we cheaply compute

\[
P(O_1O_2\ldots O_t)
\]

(How) can we cheaply compute

\[
P(q_t = S_i | O_1O_2\ldots O_t)
\]
In our example

\[ \alpha_t(i) = P(O_1 O_2 \ldots O_t \land q_t = S_i | \lambda ) \]
\[ \alpha_1(i) = b_i(O_1) \pi_i \]
\[ \alpha_{t+1}(j) = \sum_i a_{ij} b_j(O_{t+1}) \alpha_t(i) \]

**WE SAW** \( O_1 O_2 O_3 = X X Z \)

\[ \alpha_1(1) = \frac{1}{4} \quad \alpha_1(2) = 0 \quad \alpha_1(3) = 0 \]
\[ \alpha_2(1) = 0 \quad \alpha_2(2) = 0 \quad \alpha_2(3) = \frac{1}{12} \]
\[ \alpha_3(1) = 0 \quad \alpha_3(2) = \frac{1}{72} \quad \alpha_3(3) = \frac{1}{72} \]

**Easy Question**

We can cheaply compute

\[ \alpha_t(i) = P(O_1 O_2 \ldots O_t \land q_t = S_i) \]

(How) can we cheaply compute

\[ P(O_1 O_2 \ldots O_t) \]

(How) can we cheaply compute

\[ P(q_t = S_i | O_1 O_2 \ldots O_t) \]
Easy Question

We can cheaply compute
\[ \alpha_t(i) = P(O_1O_2...O_t \land q_t = S_i) \]
(How) can we cheaply compute
\[ P(O_1O_2...O_t) \]
\[ \sum_{i=1}^{N} \alpha_t(i) \]
(How) can we cheaply compute
\[ P(q_t = S_i | O_1O_2...O_t) \]
\[ \frac{\alpha_t(i)}{\sum_{j=1}^{N} \alpha_t(j)} \]

Most probable path given observations

What's most probable path given \( O_1O_2...O_T \), i.e.
What is \( \arg\max_Q P(Q | O_1O_2...O_T) \)?
Slow, stupid answer:
\[ \arg\max_Q P(Q | O_1O_2...O_T) \]
\[ = \arg\max_Q \frac{P(O_1O_2...O_T | Q)P(Q)}{P(O_1O_2...O_T)} \]
\[ = \arg\max_Q P(O_1O_2...O_T | Q)P(Q) \]
Easy Question

We can cheaply compute
\[ \alpha_t(i) = P(O_1 O_2 \ldots O_t \land q_t = S_i) \]
(How) can we cheaply compute
\[ P(O_1 O_2 \ldots O_t) \]
\[ \sum_{i=1}^{N} \alpha_t(i) \]

(How) can we cheaply compute
\[ P(q_t = S_i | O_1 O_2 \ldots O_t) \]
\[ \frac{\alpha_t(i)}{\sum_{j=1}^{N} \alpha_t(j)} \]

Most probable path given observations

What's most probable path given \( O_1 O_2 \ldots O_T \), i.e.
What is \( \text{argmax}_Q P(Q | O_1 O_2 \ldots O_T) \)?

Slow, stupid answer:
\[ \text{argmax}_Q P(Q | O_1 O_2 \ldots O_T) \]
\[ = \text{argmax}_Q \frac{P(O_1 O_2 \ldots O_T | Q) P(Q)}{P(O_1 O_2 \ldots O_T)} \]
\[ = \text{argmax}_Q P(O_1 O_2 \ldots O_T | Q) P(Q) \]
**Efficient MPP computation**

We're going to compute the following variables:

\[ \delta_t(i) = \max_{q_1 q_2 \ldots q_{t-1}} P(q_1 q_2 \ldots q_{t-1} \land q_t = S_i \land O_1 \ldots O_l) \]

= The Probability of the path of Length t-1 with the maximum chance of doing all these things:

...OCCURRING

and

...ENDING UP IN STATE \( S_i \)

and

...PRODUCING OUTPUT \( O_1 \ldots O_l \)

**DEFINE:** \( mpp_t(i) = \) that path

**So:** \( \delta_t(i) = \text{Prob}(mpp_t(i)) \)

---

**The Viterbi Algorithm**

\[ \delta_t(i) = \max_{q_1 q_2 \ldots q_{t-1}} \text{arg max}_{q_t} P(q_1 q_2 \ldots q_{t-1} \land q_t = S_i \land O_1 O_2 \ldots O_l) \]

\[ mpp_t(i) = \text{arg max}_{q_1 q_2 \ldots q_{t-1}} P(q_1 q_2 \ldots q_{t-1} \land q_t = S_i \land O_1 O_2 \ldots O_l) \]

\[ \delta_t(i) = \text{one choice } P(q_t = S_i \land O_1) \]

= \( p(q_t = S_i) p(O_1 | q_t = S_i) \)

= \( \pi_i b_i (O_1) \)

Now, suppose we have all the \( \delta_t(i)'s \) and \( mpp_t(i)'s \) for all \( i \).

**HOW TO GET** \( \delta_{t+1}(i) \) and \( mpp_{t+1}(i) \)?

mpp_{(1)} \( \rightarrow \text{Prob} = \delta_{t}(1) \)

mpp_{(2)} \( \rightarrow \text{Prob} = \delta_{t}(2) \)

\vdots

mpp_{(N)} \( \rightarrow \text{Prob} = \delta_{t}(N) \)

q_{l+1} \rightarrow S_j
Efficient MPP computation

We’re going to compute the following variables:

\[ \delta_t(i) = \max_{q_1q_2\ldots q_{t-1}} P(q_1q_2\ldots q_{t-1} \land q_t = S_i \land O_1 \ldots O_t) \]

= The Probability of the path of Length t-1 with the maximum chance of doing all these things:

…OCCURRING
and
…ENDING UP IN STATE S_i
and
…PRODUCING OUTPUT O_1\ldots O_t

DEFINE: \( mpp_t(i) = \) that path

So: \( \delta_t(i) = \text{Prob}(mpp_t(i)) \)

The Viterbi Algorithm

\[ \delta_t(i) = \max_{q_1q_2\ldots q_{t-1}} P(q_1q_2\ldots q_{t-1} \land q_t = S_i \land O_1 \ldots O_t) \]

\[ \arg\max mpp_t(i) = \max_{q_1q_2\ldots q_{t-1}} P(q_1q_2\ldots q_{t-1} \land q_t = S_i \land O_1 \ldots O_t) \]

\[ \delta_t(i) = \text{one choice } P(q_t = S_i \land O_t) \]
\[ = P(q_t = S_i) P(O_t | q_t = S_i) \]
\[ = \pi_t b_t(O_t) \]

Now, suppose we have all the \( \delta_t(i) \)'s and \( mpp_t(i) \)'s for all i.

HOW TO GET \( \delta_{t+1}(i) \) and \( mpp_{t+1}(i) \)?

\[ mpp_t(1) \rightarrow \text{Prob}=\delta_t(1) \rightarrow S_1 \]
\[ mpp_t(2) \rightarrow \text{Prob}=\delta_t(2) \rightarrow S_2 \]
\[ \vdots \]
\[ mpp_t(N) \rightarrow \text{Prob}=\delta_t(N) \rightarrow S_N \]

q_{t+1}
The Viterbi Algorithm

The most probable path with last two states \( S_i, S_j \) is the most probable path to \( S_i \), followed by transition \( S_i \rightarrow S_j \).

What is the probability of that path?

\[
\delta(i) \times P(S_i \rightarrow S_j \land O_{t+1} | \lambda) = \delta(i) \cdot a_{ij} \cdot b_j(O_{t+1})
\]

So the most probable path to \( S_j \) has \( S_{i^*} \) as its penultimate state where \( i^* = \arg\max_i \delta(i) \cdot a_{ij} \cdot b_j(O_{t+1}) \).
The Viterbi Algorithm

The most probable path with last two states $S_i$ $S_j$

is the most probable path to $S_i$, followed by transition $S_i \rightarrow S_j$.

What is the prob of that path?

$$\delta_t(i) \times P(S_i \rightarrow S_j \land O_{t+1} \mid \lambda)$$

$$= \delta_t(i) a_{ij} b_j(O_{t+1})$$

SO The most probable path to $S_j$ has $S_{i^*}$ as its penultimate state

where $i^* = \text{argmax}_i \delta_t(i) a_{ij} b_j(O_{t+1})$.
The Viterbi Algorithm

The most prob path with last two states $S_i, S_j$ is the most prob path to $S_i$, followed by transition $S_i \rightarrow S_j$

What is the prob of that path?

$\delta_t(i) \times P(S_i \rightarrow S_j \land O_{t+1})$

$= \delta_t(i) a_{ij} b_j(O_{t+1})$

SO The most probable $S_j$ as its penultimate state

where $i^* = \arg \max_i \delta_t(i) a_{ij} b_j(O_{t+1})$

Summary:

$\delta_{t+1}(j) = \delta_t(i^*) a_{ij} b_j(O_{t+1})$

$mpp_{t+1}(j) = mpp_{t+1}(i^*)S_{i^*}$

$\{ \text{with } i^* \text{ defined to the left} \}$

What’s Viterbi used for?

Classic Example
Speech recognition:

Signal $\rightarrow$ words

HMM $\rightarrow$ observable is signal

$\rightarrow$ Hidden state is part of word formation

What is the most probable word given this signal?

UTTERLY GROSS SIMPLIFICATION

In practice: many levels of inference; not one big jump.
The Viterbi Algorithm

The most probable path with last two states $S_i, S_j$ is
the most probable path to $S_i$, followed by transition $S_i \rightarrow S_j$.

What is the prob of that path?

$\delta_t(i) \times P(S_i \rightarrow S_j \land O_{t+1})$

$= \delta_t(i) a_{ij} b_j(O_{t+1})$

So the most probable $S_i$ as its penultimate state
where $i^* = \text{argmax}_i \delta_t(i) a_{ij} b_j(O_{t+1})$

Summary:

$\delta_{t+1}(j) = \delta_t(i^*) a_{ij} b_j(O_{t+1})$

$mpp_{t+1}(j) = mpp_{t+1}(i^*)S_i$

What’s Viterbi used for?

Classic Example

Speech recognition:

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What is the most probable word given this signal?

UTTERLY GROSS SIMPLIFICATION

In practice: many levels of inference; not one big jump.
HMMs are used and useful

But how do you design an HMM?

Occasionally, (e.g. in our robot example) it is reasonable to deduce the HMM from first principles.

But usually, especially in Speech or Genetics, it is better to infer it from large amounts of data. $O_1 O_2 .. O_T$ with a big “T”.

Observations previously in lecture

$O_1 O_2 .. O_T$

Observations in the next bit

$O_1 O_2 .. O_T$

Inferring an HMM

Remember, we’ve been doing things like

$$P(O_1 O_2 .. O_T | \lambda)$$

That “$\lambda$” is the notation for our HMM parameters.

Now we have some observations and we want to estimate $\lambda$ from them.

AS USUAL: We could use

(i) MAX LIKELIHOOD $\lambda = \text{argmax } P(O_1 .. O_T | \lambda)$

(ii) BAYES

Work out $P(\lambda | O_1 .. O_T)$

and then take $E[\lambda]$ or max $P(\lambda | O_1 .. O_T)$
HMMs are used and useful

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Inferring an HMM

Remember, we’ve been doing things like

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Now we have some observations and we want to estimate $\lambda$ from them.

AS USUAL: We could use

(i) MAX LIKELIHOOD

$$\lambda = \arg \max_{\lambda} P(O_1 .. O_T | \lambda)$$

(ii) BAYES

Work out $P(\lambda | O_1 .. O_T)$

and then take $E[\lambda]$ or $\max_{\lambda} P(\lambda | O_1 .. O_T)$.
Max likelihood HMM estimation

Define
\[ \gamma_t(i) = P(q_t = S_i | O_1O_2...O_T, \lambda) \]
\[ \epsilon_t(i,j) = P(q_t = S_i \land q_{t+1} = S_j | O_1O_2...O_T, \lambda) \]

\( \gamma_t(i) \) and \( \epsilon_t(i,j) \) can be computed efficiently \( \forall i,j,t \)
(Details in Rabiner paper)

\[ \sum_{t=1}^{T-1} \gamma_t(i) = \text{Expected number of transitions out of state } i \text{ during the path} \]
\[ \sum_{t=1}^{T-1} \epsilon_t(i,j) = \text{Expected number of transitions from state } i \text{ to state } j \text{ during the path} \]

HMM estimation

\[ \gamma_t(i) = P(q_t = S_i | O_1O_2...O_T, \lambda) \]
\[ \epsilon_t(i,j) = P(q_t = S_i \land q_{t+1} = S_j | O_1O_2...O_T, \lambda) \]

\[ \sum_{t=1}^{T-1} \gamma_t(i) = \text{expected number of transitions out of state } i \text{ during path} \]
\[ \sum_{t=1}^{T-1} \epsilon_t(i,j) = \text{expected number of transitions out of } i \text{ and into } j \text{ during path} \]

Notice \[ \frac{\sum \epsilon_t(i,j)}{\sum \gamma_t(i)} = \frac{\text{(expected frequency)}}{\text{(expected frequency)}} \]

= Estimate of \( \text{Prob}(\text{Next state } S_j | \text{This state } S_i) \)

We can re-estimate
\[ a_{i,j} \leftarrow \frac{\sum \epsilon_t(i,j)}{\sum \gamma_t(i)} \]

We can also re-estimate
\[ b_j(O_k) \leftarrow \cdots \]
(See Rabiner)
Max likelihood HMM estimation

Define

\[ \gamma_t(i) = P(q_t = S_i \mid O_1O_2\ldots O_T, \lambda) \]
\[ \varepsilon_t(i,j) = P(q_t = S_i \land q_{t+1} = S_j \mid O_1O_2\ldots O_T, \lambda) \]

\( \gamma_t(i) \) and \( \varepsilon_t(i,j) \) can be computed efficiently \( \forall i,j,t \)

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\[ \sum_{t=1}^{T-1} \gamma_t(i) = \text{Expected number of transitions out of state } i \text{ during the path} \]

\[ \sum_{t=1}^{T-1} \varepsilon_t(i,j) = \text{Expected number of transitions from state } i \text{ to state } j \text{ during the path} \]
We want $a_{ij}^{\text{new}}$ = new estimate of $P(q_{t+1} = s_j \mid q_t = s_i)$

$$= \frac{\text{Expected # transitions } i \rightarrow j \mid \lambda_k^{\text{old}}, O_1, O_2, \ldots O_T}{\sum_{k=1}^{N} \text{Expected # transitions } i \rightarrow k \mid \lambda_k^{\text{old}}, O_1, O_2, \ldots O_T}$$
We want $a_{ij}^{\text{new}} = \text{new estimate of } P(q_{t+1} = s_j \mid q_t = s_i)$

$= \frac{\text{Expected } \# \text{ transitions } i \rightarrow j \mid \lambda^\text{old}, O_1, O_2, \ldots O_T}{\sum_{k=1}^{N} \text{Expected } \# \text{ transitions } i \rightarrow k \mid \lambda^\text{old}, O_1, O_2, \ldots O_T}$
We want $a_{ij}^{\text{new}} = \text{new estimate of } P(q_{t+1} = s_j \mid q_t = s_i)$

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$$= \frac{\sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}{\sum_{k=1}^{N} \sum_{t=1}^{T} P(q_{t+1} = s_k, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}$$

$$= \frac{\sum_{k=1}^{N} \sum_{t=1}^{T} P(q_{t+1} = s_k, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}{\sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}$$
We want \( a_{ij}^{\text{new}} = \text{new estimate of } P(q_{t+1} = s_j \mid q_t = s_i) \)

\[
= \frac{\text{Expected # transitions } i \rightarrow j \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T}{\sum_{k=1}^{N} \text{Expected # transitions } i \rightarrow k \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T}
\]

\[
= \frac{\sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}{\sum_{k=1}^{N} \sum_{t=1}^{T} P(q_{t+1} = s_k, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}
\]

\[
= \frac{S_{ij}}{\sum_{k=1}^{N} S_{ik}} \quad \text{where } S_{ij} = \sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i, O_1, \cdots O_T \mid \lambda^{\text{old}})
\]

\[
= \text{What?}
\]
We want $a_{ij}^{\text{new}} = \text{new estimate of } P(q_{t+1} = s_j \mid q_t = s_i)$

$$= \frac{\text{Expected } # \text{ transitions } i \rightarrow j \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T}{\sum_{k=1}^{N} \text{Expected } # \text{ transitions } i \rightarrow k \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T}$$

$$= \frac{\sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T)}{\sum_{k=1}^{N} \sum_{t=1}^{T} P(q_{t+1} = s_k, q_t = s_i \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T)}$$

$$= \frac{S_{ij}}{\sum_{k=1}^{N} S_{ik}} \text{ where } S_{ij} = \sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i, O_1, \cdots O_T \mid \lambda_{\text{old}})$$

$$= a_{ij} \sum_{t=1}^{T} \alpha_t(i) \beta_{t+1}(j) b_j(O_{t+1})$$

We want $a_{ij}^{\text{new}} = \frac{S_{ij}}{\sum_{k=1}^{N} S_{ik}} \text{ where } S_{ij} = a_{ij} \sum_{t=1}^{T} \alpha_t(i) \beta_{t+1}(j) b_j(O_{t+1})$
We want $a_{ij}^{\text{new}} =$ new estimate of $P(q_{t+1} = s_j \mid q_t = s_i)$

$$= \frac{\text{Expected # transitions } i \to j \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T}{\sum_{k=1}^{N} \text{Expected # transitions } i \to k \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T}$$

$$= \frac{\sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T)}{\sum_{k=1}^{N} \sum_{t=1}^{T} P(q_{t+1} = s_k, q_t = s_i \mid \lambda_{\text{old}}, O_1, O_2, \cdots O_T)}$$

$$= \frac{S_{ij}}{\sum_{k=1}^{N} S_{ik}} \text{ where } S_{ij} = \sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i, O_1, \cdots O_T \mid \lambda_{\text{old}})$$

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We want $a_{ij}^{\text{new}} = S_{ij} / \sum_{k=1}^{N} S_{ik}$ where $S_{ij} = a_{ij} \sum_{t=1}^{T} \alpha_t(i) \beta_{t+1}(j) b_j(O_{t+1})$
We want \( a_{ij}^{\text{new}} = S_{ij} \sqrt{\sum_{k=1}^{N} S_{ik}} \) where \( S_{ij} = a_{ij} \sum_{t=1}^{T} \alpha_t(i) \beta_{t+1}(j) b_j(O_{t+1}) \).

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If we knew \( \lambda \) we could estimate EXPECTATIONS of quantities such as
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Roll on the EM Algorithm…
We want \( a_{ij}^{\text{new}} = S_{ij} \sum_{k=1}^{N} S_{ik} \) where \( S_{ij} = a_{ij} \sum_{t=1}^{T} \alpha_{i}(i) \beta_{t+1}(j) b_{j}(O_{t+1}) \)

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Roll on the EM Algorithm...
EM 4 HMMs

1. Get your observations \( O_1 \ldots O_T \)
2. Guess your first \( \lambda \) estimate \( \lambda(0) \), \( k=0 \)
3. \( k = k+1 \)
4. Given \( O_1 \ldots O_T \), \( \lambda(k) \) compute
   \( \gamma_{t}(i) \), \( \epsilon_{t}(i,j) \) \( \forall \ 1 \leq t \leq T, \ \forall \ 1 \leq i \leq N, \ \forall \ 1 \leq j \leq N \)
5. Compute expected freq. of state \( i \), and expected freq. \( i \rightarrow j \)
6. Compute new estimates of \( a_{ij}, b_{j}(k), \pi_{i} \) accordingly. Call them \( \lambda(k+1) \)
7. Goto 3, unless converged.
   • Also known (for the HMM case) as the BAUM-WELCH algorithm.

Bad News

• There are lots of local minima

Good News

• The local minima are usually adequate models of the data.

Notice

• EM does not estimate the number of states. That must be given.
• Often, HMMs are forced to have some links with zero probability. This is done by setting \( a_{ij}=0 \) in initial estimate \( \lambda(0) \)
• Easy extension of everything seen today: HMMs with real valued outputs
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2. Guess your first $\lambda$ estimate $\lambda(0)$, $k=0$
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   $\gamma_t(i), \epsilon_t(i,j), \forall 1 \leq t \leq T, \forall 1 \leq i \leq N, \forall 1 \leq j \leq N$
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What You Should Know

• What is an HMM?
• Computing (and defining) $\alpha_t(i)$
• The Viterbi algorithm
• Outline of the EM algorithm
• To be very happy with the kind of maths and analysis needed for HMMs
• Fairly thorough reading of Rabiner* up to page 266* [Up to but not including “IV. Types of HMMs”].

http://ieeexplore.ieee.org/xpl/servlet/FliedServlet?arnumber=18626
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Trade-off between too few states (inadequately modeling the structure in the data) and too many (fitting the noise).
Thus #states is a regularization parameter.
Blah blah blah... bias variance tradeoff...blah blah...cross-validation...blah blah....AIC, BIC....blah blah (same ol' same ol')

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DON’T PANIC: starts on p. 257.