Lecture 9b: Low Reynolds Number Flows

Summary of High/Low Reynolds Flow

Conservation of mass leads to continuity equation: \( v_1 A_1 = v_2 A_2 \) for incompressible flows
*Always applicable!*

**High Reynolds Number flow (last time):**

1. Bernoulli equation relates \( P, v, \) and \( h: \)
   
   \[ p_1 + \frac{1}{2} \rho v_1^2 + \rho g h_1 = \text{constant} \rightarrow \text{conservation of energy/Volume} \]

   Note: All turbulent flows are High Re but not all High Re number flows are turbulent

2. Bernoulli application: Venturi effect

   \[ p_1 \cdot p_2 = \frac{1}{2} \rho (v_2^2 - v_1^2) \]

**Low Reynolds Number flow (TODAY!):**

3. **Newton’s law of viscosity** relates shear stress \( \tau \) to fluid velocity \( v: \)
   
   \[ \tau = \frac{F}{A} = \eta \frac{v}{l} \]

   (Example: force needed to spread a layer of honey of thickness \( l \) at speed \( v \) with knife of area \( A \))

   Low Reynolds number flows are reversible: small organisms swim differently than we do!

4. The **Stokes equation** \( F_{\text{drag}} = 6 \pi \eta v R \) tells us the drag force on a sphere of radius \( R \).

5. The **Poiseuille equation** \( Q = \frac{\pi \Delta P}{8 \eta l} R^4 \) describes flow rate in a pipe of length \( l \) and radius \( R \).
Learning objectives: After this lecture, you will be able to:

1. Qualitatively and quantitatively explain the meaning of viscosity.
2. Understand the qualitative differences between high Reynolds number flows and low Reynolds number flows and the role viscosity plays in both.
3. Explain and use Newton’s law of viscosity.
4. Explain that in low Reynolds number flows the force applied to a fluid is used to overcome viscous forces, not to accelerate the fluid.
5. Appreciate that low Reynolds number flows are reversible and that small organisms swim differently than we do!
6. To use the Stokes equation and explain that it tells us the drag force on a sphere in a low Re flow.
7. To use the Poiseuille equation and explain how it relates the velocity $v$ of a laminar flow inside a pipe to the pressure drop.
L9b: Am I getting it? #1

1. Air flows upward toward, and around, a beach ball. The air in the middle is flowing faster than the air on the sides (see figure). If you nudge the ball slightly to the right as the air flows around it, where will the pressure be highest?
   a) To the left of the ball
   b) To the right of the ball
   c) Same pressure to left and right of the ball

2. If you put topspin on a baseball or tennis ball that is moving horizontally, will the ball hit the ground earlier or later (compared to a case where you didn’t put topspin on it)?
   a) Earlier
   b) Later
   c) Same time in both cases

3. There is a steady 15 mph wind parallel to the ground, traveling from the back to the front of the house (ignore the presence of trees). The wind flows over the top of the house.

   At which point is the pressure lowest?
   a) Back of the house
   b) Above the house (rooftop)
   c) Front of the house

4. Two tanks are filled with a fluid of unknown viscosity. Tank 1 is drained through a tube of radius \( R \), while tank 2 is drained through 16 smaller tubes, each with radius \( R/2 \). Which tank will empty first? (not on canvas; write prediction here)
Activity 1: Newton’s law of viscosity

Newton’s law of viscosity: It tells us how much force is needed to make a viscous fluid flow

\[ \tau = \frac{F}{A} = \eta \, \frac{v}{l} \quad \text{OR} \quad F = \eta \, \frac{A \, v}{l} , \]

where: \( \tau = \frac{F}{A} \) = Shear Stress, \( \frac{v}{l} \) = Shear Rate.

Shear a fluid between parallel plates

Force or shear stress \( \tau \) required to slide top plate at speed \( V \)

In a Newtonian fluid, \( \eta \) is independent of \( v \)

\( \eta_{\text{air}} \approx 2 \times 10^{-5} \, \text{Pa} \cdot \text{s} \)
\( \eta_{\text{water}} \approx 10^{-3} \, \text{Pa} \cdot \text{s} \)
\( \eta_{\text{peanut butter}} \approx 10^{2} \, \text{Pa} \cdot \text{s} \)

1. Water is trapped between two surfaces (see figure). For the stationary lower surface, what is the force on a small patch of area 1 cm\(^2\) (0.0001 m\(^2\))?

\[ F = \eta \, \frac{A \, v}{l} = 10^{3} \, \text{Pa} \cdot \text{s} \, \frac{10^{-4} \, \text{m}^2 \cdot 0.1 \, \text{m/s}}{0.01 \, \text{m}} = 10^{-6} \, \text{N} \]

2. What is the force to slide the top slice of bread \((A = 0.01 \, \text{m}^2)\) of a peanut butter sandwich with a 1 mm thick layer of peanut butter, at \( v = 10 \, \text{cm/s} \)? \( \eta_{\text{peanut butter}} \approx 10^{2} \, \text{Pa} \cdot \text{s} \)

\[ F = \eta \, \frac{A \, v}{l} = 10^{2} \, \text{Pa} \cdot \text{s} \, \frac{0.01 \, \text{m}^2 \cdot 0.1 \, \text{m/s}}{10^{-3} \, \text{m}} = 100 \, \text{N} \\ 22 \, \text{lbs} \]

3. When paddling a canoe upstream, it is wisest to travel as near to the shore as possible. When canoeing downstream, it may be best to stay near the middle. Explain why.
Qualitative features of low Re flows

- laminar, reversible

Using the Reynolds number to model flows:

1) Demo: mixing ink in a viscous fluid

2) Video: from a movie by G.I. Taylor

Low Re flows are reversible: small organisms swim differently than we do!
Activity 2
Two types of low \( Re \) flows: (1) external and (2) internal

<table>
<thead>
<tr>
<th>Translation of an object</th>
<th>Flow through a pipe</th>
</tr>
</thead>
<tbody>
<tr>
<td>- What force is needed to drag the sphere at \textit{steady speed} ( v )?</td>
<td>- What is the velocity distribution in the pipe?</td>
</tr>
<tr>
<td></td>
<td>- What pressure is needed to pump fluid through a pipe of radius ( R ) and length ( L )</td>
</tr>
<tr>
<td></td>
<td>- What flow rate (volume/time) occurs?</td>
</tr>
</tbody>
</table>

\textbf{Stokes equation} \hspace{1cm} \textbf{Poiseuille equation}

1. The Stokes equation for \( F_{\text{drag}} \) depends on \( \eta \) (\textit{viscosity}), Radius \( R \), and speed \( v \).

   (a) The dependence of \( F_{\text{drag}} \) on \( v \) is \textit{proportional} OR \textit{inversely proportional}?

   (b) The dependence of \( F_{\text{drag}} \) on \( R \) is \textit{proportional} OR \textit{inversely proportional}?

   (c) The dependence of \( F_{\text{drag}} \) on \( \eta \) is \textit{proportional} OR \textit{inversely proportional}?

2. The Poiseuille equation for the flow rate \( Q \) depends on \( \eta \) (\textit{viscosity}), \( \Delta P \) (pressure difference driving the flow), Radius \( R \), and length of pipe \( l \).

   (a) The dependence of \( Q \) on \( \eta \) is \textit{proportional} OR \textit{inversely proportional}?

   (b) The dependence of \( Q \) on \( R \) is \textit{proportional} OR \textit{inversely proportional}?

   (c) The dependence of \( Q \) on \( \Delta P \) is \textit{proportional} OR \textit{inversely proportional}?

   (d) The dependence of \( Q \) on \( l \) is \textit{proportional} OR \textit{inversely proportional}?
Activity 3A: Stokes and Poiseuille equations

Stokes equation: $F_{\text{drag}} = 6\pi \eta v R$

Poiseuille equation: $Q = \frac{\pi \Delta P}{\eta l} R^4$

Parabolic flow profile

Stokes drag on a sphere has been an important aspect of at least THREE Nobel Prizes (1923, 1926 and 1926): 1) Millikan’s oil drop experiment, 2) Perrin’s experiments on Brownian motion, 3) Svedberg’s study of protein separation using an ultracentrifuge.

This is important for homes, chemical power plants, respiration, blood flow, water transport in plants, oil recovery, etc…

Remember that we are in the low $Re$ number regime (viscous dominated) with laminar flow.

1. Draw the streamlines for the case of low and high Reynolds flow for a fluid going through a pipe (guess the shape of the streamlines): Higher velocity = streamlines are closer together

2. Draw the streamlines for the case of low and high Reynolds flow for a solid sphere moving through a fluid at speed $v$ (guess the shape of the streamlines): Higher velocity = streamlines are closer together
Activity 3B: Stokes and Poiseuille equations

1. Application of Poiseuille equation: \[ Q = \frac{\pi \Delta P}{8 \eta l} R^4 \]

Today you saw a demonstration of Poiseuille’s Law in which water flowed out of an open tank through a long narrow tube. As shown in the diagram, the water level in the tank is \( h = 10 \text{ cm} \) above the opening at the bottom of the tank, the cylindrical tube has a length of \( L = 1 \text{ m} \) and an interior radius of \( R = 1 \text{ mm} \), and the tube is a height of \( H = 1 \text{ m} \) above the ground:

![Diagram of water flow](image)

a) What is the flow rate \( Q \) in the cylindrical tube? (Water has a density \( \rho = 10^3 \text{ kg m}^{-3} \) and a viscosity \( \eta = 10^{-3} \text{ kg m}^{-1} \text{s}^{-1} \).)

\[ Q = \frac{\pi \Delta P}{8 \eta l} R^4 \]
\[ \Delta P = P_1 - P_2 = \rho g h = \rho g \frac{L}{2} \]
\[ Q = \frac{\pi \rho g h R^4}{8 \eta l} \approx 4 \times 10^{-7} \text{ m}^3 / \text{s} \]

b) If you replaced the water with a more viscous fluid, the distance \( d \) would (circle one):

- increase
- remain the same
- decrease

2. A blood vessel develops plaque deposits that reduce the radius by a factor of 2. How much does the volumetric flow rate \( Q \) (volume/time) of blood change as a result? *I dare you to go eat a high cholesterol lunch after calculating this

\[ R_{\text{new}} = \frac{R}{2} \]

\[ Q_{\text{new}} = \frac{\pi \Delta P}{8 \eta l} \left( \frac{R}{2} \right)^4 = \left( \frac{1}{16} \right) \frac{\pi \Delta P}{8 \eta l} R^4 = \frac{1}{16} Q \]