Problem Set 10

1. Particle decays. Recall that the decay rate is given by the general formula

\[ d\Gamma = \frac{1}{2E_1} |\mathcal{M}|^2 \frac{d^3 p_2}{(2\pi)^3 2E_2} \cdot \frac{d^3 p_n}{(2\pi)^3 2E_n} \left(2\pi\right)^4 \delta^4(p_1 - p_2 - \cdots - p_n) \]  

\[ \Gamma(\phi \rightarrow e^+ + e^-) = \frac{\sqrt{1 - 4x^2}}{16\pi m_\phi} |\mathcal{M}|^2, \quad x = \frac{m_e}{m_\phi} \] 

a) Evaluate the phase space integrals for $1 \rightarrow 2$ decays. Show that the total rate is:

\[ \Gamma(\phi \rightarrow e^+ + e^-) = \frac{1}{4} \frac{\gamma}{x^2} \] 

b) Evaluate $\Gamma$ for a particle $\phi$ of mass $m_\phi$ decaying to $e^+ e^-$ of mass $m_e$ if

i. $\phi$ is a scalar, with interaction $g_{\phi\tilde{\psi}\psi}$

ii. $\phi$ is a a pseudoscalar, with interaction $ig_{\phi\tilde{\psi}\gamma_5\psi}$

iii. $\phi$ is a vector, with interaction $g_{\phi\tilde{\psi}\gamma^\mu\psi}$

iv. $\phi$ is an axial vector, with interaction $ig_{\phi\tilde{\psi}\gamma^\mu\gamma_5\psi}$

c) News flash! The LHC experiment, on which Professors Franklin, Huth and Morii work, has just discovered a new particle which decays only to leptons ($\tau^+, \mu^+$ and $e^-$) whose mass is around 4 GeV. About 25% of the time it decays to $\tau^+ \tau^-$. What spin and parity might this particle have?

2. Professor Guennette is working on an experiment called DUNE which attempts to measure masses and other properties of neutrinos. Of the strong, weak, and electromagnetic force, neutrinos only interact with the weak force, so they are very hard to see. As we saw in the last problem set, neutrino interactions violate parity: the left-handed neutrino couples to the $W$ and $Z$ bosons. No one has ever seen a right-handed neutrino, and we do not know if right-handed neutrinos exist. In 1998, neutrino mass was discovered. How can neutrinos have mass if they are only left-handed?

a) One way to give neutrinos mass is to imagine that there exist right-handed neutrinos which have no interactions. Such particles are called sterile neutrinos. Then one can write the kinetic Lagrangian for the neutrinos as

\[ \mathcal{L}_{\text{kin}} = i\nu_L^\dagger \sigma^\nu \partial_\nu \nu_L + i\nu_R^\dagger \sigma^\nu \partial_\nu \nu_R - m_L\nu_L^\dagger \nu_L + m_R\nu_R^\dagger \nu_R + i\frac{M}{2}(\nu_R^\dagger \sigma_2 \nu_L - \nu_L^\dagger \sigma_2 \nu_R) \] 

Here, $\nu_L$ is a left-handed ($\frac{1}{2}, 0$) 2-component Weyl spinor and $\nu_R$ is a right handed ($0, \frac{1}{2}$) Weyl spinor. Note that there are two mass terms: a Dirac mass $m$, like for the electron, and a Majorana mass, $M$.

Show that this Lagrangian is Lorentz invariant and that $\chi_L \equiv i\sigma_2 \nu_R^\dagger$ transforms as a left-handed spinor under the Lorentz group, so that it can mix with $\nu_L$.

b) What are the mass eigenstates? That is, find linear combinations $\psi_1$ and $\psi_2$ of $\chi_L$ and $\nu_L$ which satisfy the Klein Gordon equation $(\Box + m^2)\psi = 0$. What are $m_i$?

c) Suppose $M \gg m$. For example, $M = 10^{15}$ GeV and $m = 100$ GeV. What are the masses of the physical particles (mass eigenvalues)? The fact that as $M$ goes up, the physical masses go down, inspired the name see-saw mechanism for this neutrino mass arrangement. What other choice of $M$ and $m$ would give the same spectrum of observed particles (i.e. particles less than $\sim 1$ TeV)?

d) The left-handed neutrino couples to the $Z$ boson and also to the electron through the $W$ boson. The $W$ boson also couples the neutron and proton. The relevant part for the weak-force Lagrangian is

\[ \mathcal{L}_{\text{weak}} = g_W (\nu_L^\dagger W^\nu e_L + e_L^\dagger W \nu_L) + g_Z (\nu_L^\dagger Z \nu_L) + g_W (n W^\nu p + \bar{n} W^\nu \bar{p}) \]
Using these interactions, draw a Feynman diagram for neutrino-less double-$\beta$ decay, in which two neutrons decay to two protons and two electrons.

e) Which of the terms in $L_{\text{kin}}$ and $L_{\text{weak}}$ respect a global symmetry (lepton number) under which $\nu_L \to e^\theta \nu_L$, $\nu_R \to e^\theta \nu_R$ and $e_L \to e^\theta e_L$? Define arrows on the $e$ and $\nu$ lines to respect lepton number flow (instead of charge flow). Show that you cannot connect the arrows on your diagram without violating lepton number. Does this imply that neutrinoless double $\beta$ decay can tell if the light neutrino has a Majorana mass (i.e. distinguish the two mass mechanisms with identical light masses you found in part c)?

3. Professor Doyle is working on experiments to measure electric and magnetic dipole moments of various fundamental particles. We showed that the electron has a magnetic dipole moment, of order $\mu_B = \frac{e^2}{2m_e}$, by squaring the Dirac equation. An additional magnetic moment could come from an interaction of the form $B = i F_{\mu\nu} \bar{\psi} [\gamma^\mu, \gamma^\nu] \psi$ in the Lagrangian. An electric dipole moment (EDM) corresponds to a term of the form $E = F_{\mu\nu} \bar{\psi} \gamma_5 [\gamma^\mu, \gamma^\nu] \psi$.

a) Expand the contribution of the electric dipole term to the Dirac equation in terms of electric and magnetic fields, to show that it does in fact give an electric dipole moment.

b) Consider the discrete symmetries charge conjugation ($C$), parity ($P$) and time-reversal ($T$). On spinors and gauge fields, these act as (in the Weyl basis for $\gamma_\mu$) [cf. Chapter 11 for more details]

\begin{equation}
C: \psi \rightarrow -i \gamma_2 \psi^*, \quad A_\mu \rightarrow -A_\mu \\
P: \psi(t, \vec{x}) \rightarrow \gamma_0 \psi(t, -\vec{x}), \quad A_0(t, \vec{x}) \rightarrow -A_0(t, -\vec{x}), \quad A_i(t, \vec{x}) \rightarrow -A_i(t, -\vec{x}) \\
T: \psi(-t, \vec{x}) \rightarrow \gamma_1 \gamma_3 \psi(-t, -\vec{x}), \quad A_0(t, \vec{x}) \rightarrow A_0(-t, -\vec{x}), \quad A_i(t, \vec{x}) \rightarrow -A_i(-t, -\vec{x})
\end{equation}

in addition, $T$ sends $i \rightarrow -i$ in the whole Lagrangian, so it sends $\gamma_2 \rightarrow -\gamma_2$, but $\gamma_{0,1,3} \rightarrow \gamma_{0,1,3}$. $C$, $P$ and $T$ are all symmetries of QED. The combined operation $CPT$ is a symmetry of any local Lorentz invariant quantum field theory.

Which of the symmetries $C$, $P$ or $T$ are respected by the magnetic dipole moment operator, $B$ and the EDM operator, $E$?

c) It turns out that, $C$, $P$ and $T$ are all separately violated in the Standard Model, even though they are preserved in QED (and QCD). $P$ is violated by the weak interactions, but $T$ (and $C$ and $P$) is only very weakly violated. Thus we expect, unless there is a new source of CP violation beyond the standard model, the electron, the neutron, the proton, the deuteron etc., all should have unmeasurably small (but nonzero) EDMs. Why is it ok for a molecule (like $H_2O$) or a battery to have an EDM but not the neutron (which is made up of quarks with different charges)?

4. *Casimir forces. The Casimir force was first measured conclusively by Lamoreaux in 1997. Even repulsive Casimir forces have been measured in systems with varying dielectric constants, for example, by Professor Capasso in SEAS.

a) Calculate the Casimir force in 1D using a Gaussian regulator.

b) Show that the Casimir force from the vacuum energy of fermions has the opposite sign than from bosons.

c) Some people have proposed that geckos use the Casimir force to climb walls. It is known that geckos do not use suction (like salamanders) or capillary adhesion (like some frogs). A gecko’s foot is made of a million tiny hairs called setae which terminate in spatula-shaped structures around 0.5$\mu$m wide. Use dimensional analysis and the form of the Casimir force to decide whether you think that the Casimir explanation of the gecko’s abilities could be possible. Treat this as a Fermi problem – estimate everything you don’t know. For example, you might assume that the hairs can get within molecular spacings of the wall the gecko is climbing.