Problem Set 9

1. Consider the following diagram for \( e^+e^- \rightarrow \mu^+\mu^- \) in QED

\[
\begin{array}{c}
\text{(diagram)}
\end{array}
\]

a) How many diagrams contribute at the same order in perturbation theory?

b) What is the minimal set of diagrams you need to add to this one for the sum to be gauge invariant (independent of \( \xi \))?

c) Show explicitly that the sum of diagrams in part b is gauge invariant.

2. Parity Violation.

We calculated that \( e^+e^- \rightarrow \mu^+\mu^- \) at high energy has a \( 1 + \cos^2 \theta \) angular dependence, where \( \theta \) is the angle between the \( e^- \) and \( \mu^- \) directions. This agrees with experiment, as the simulated data in Figure 1 shows. However, the angular distribution for scattering into muon neutrinos \( e^+e^- \rightarrow \nu_\mu \bar{\nu}_\mu \) is very different, as shown in the simulated data in Figure 2, where now \( \theta \) is the angle between the \( e^- \) and \( \nu_\mu \) directions:

![Figure 1](image1.png)  
![Figure 2](image2.png)

a) At low energy, the total cross section \( \sigma_{tot} \) for \( e^+e^- \rightarrow \nu_\mu \bar{\nu}_\mu \) scattering grows with energy, in contrast to the total \( e^+e^- \rightarrow \mu^+\mu^- \) cross section. Show that this is consistent with neutrino scattering being mediated by a massive vector boson, the \( Z \). Deduce how \( \sigma_{tot} \) should depend on \( E_{CM} \) for the two processes.

b) Place the neutrino in a Dirac spinor \( \psi_\nu \). There are two possible couplings we could write down for \( \nu \) to the new massive gauge boson: \( g_V \psi_\nu Z \psi_\nu + g_A \psi_\bar{\nu} Z \gamma^\mu \psi_\nu \). These are called vector and axial-vector couplings, respectively. Assume the \( Z \) couples to the electron in the same way as it couples to neutrinos. Calculate the full angular dependence for \( e^+e^- \rightarrow \nu_\mu \bar{\nu}_\mu \) as a function of \( g_V \) and \( g_A \) (you can drop fermion masses).

c) What values of \( g_V \) and \( g_A \) reproduce figure 2? Show that this choice is equivalent to the \( Z \) boson having chiral couplings: it only interacts with left-handed fields. Argue that this is evidence of parity violation, where the parity operator \( P \) is reflection in a mirror: \( \vec{x} \rightarrow -\vec{x} \).

d) An easier way to see parity violation is in \( \beta \)-decay. This is mediated by charged gauge bosons, the \( W^\pm \), that are “unified” with the \( Z \). Assuming they have the same chiral couplings as the \( Z \), draw a diagram to show that the electron coming out of \( C^{60} \rightarrow Ni^{59} + e^- + \tilde{\nu} \) will always be left-handed, independent of the spin of the Cobalt nucleus. What handedness would the positron be in anti-Cobalt decay: \( C^{60} \rightarrow Ni^{59} + e^+ + \nu? \)

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e) If you are talking to aliens on the telephone (i.e. with light only), tell them how to use nuclear $\beta$ decay to tell clockwise from counterclockwise. For this, you will need to figure out how to relate the $L$ in $\psi_L$ to “left” in the real world. You are allowed to assume all the materials on earth are available to them, including things like Cobalt, birefractive materials like calcite, and lasers.

f) If you meet those aliens, and put out your right hand to greet them, but they put out their left hand, why should you not shake? (this scenario is due to Feynman).

g) Now forget about neutrinos. Could you have the aliens distinguish right from left by actually sending them circularly polarized light, for example using polarized radio waves for your intergalactic telephone?

3. Path integrals

a) Since Grassmann numbers anticommute, $\theta_i \theta_2 \theta_2 = 0$, why doesn’t a term in the Lagrangian like $\bar{\psi}(x) \psi(x) \psi(x) \psi(x)$ automatically vanish? Would $(\bar{\psi} \psi)^5$ vanish? Would you get the same answer for $e^+ e^- \rightarrow 4 e^+ e^-$ pairs from a $(\psi \bar{\psi})^5$ term the Lagrangian in the canonical formalism and with the path integral? Justify your answer.

b) We showed that correlation functions of gauge-invariant operators come out the same if we add a term $-\frac{1}{2\xi} (\partial_{\mu} A_{\mu})^2$ to the Lagrangian. Would they come out the same if we added a term of the form $-\frac{1}{2\xi} (\partial_{\mu} A_{\mu})^4$? What about a term of the form $\xi A_{\mu}^2$?

4. To derive the Schwinger-Dyson equations for scalars in the canonical picture, we needed to use the equations $(\Box + m^2) \phi = L'_{int}[\phi]$ and $[\phi(x, t), \partial_t \phi(y, t)] = i \delta^4(\vec{x} - \vec{y})$:

a) What are the equivalent of these equations for Dirac spinors?

b) Verify the Schwinger-Dyson equation in Eq. (14.118) of the book:

\begin{align}
(\partial_{\mu} \partial_{\mu} + m^2) & \psi(x) \psi(y) \psi(z) \psi(w) = -e \langle \bar{\psi}(x) A_{\mu} \psi(y) \bar{\psi}(z) \psi(w) \rangle \\
- i \delta_{\mu \nu} & \delta^4(x - y) \psi_{\sigma}(y) \bar{\psi}_{\beta}(z) - i \delta_{\mu \nu} \delta^4(x - y) \langle \psi_{\sigma}(w) \bar{\psi}_{\beta}(z) \rangle
\end{align}

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using the canonical approach with your equations from part a) and also with the path integral.

5. Construct the states which satisfy $\phi(x) | \Phi \rangle = \Phi(x) | \Phi \rangle$ explicitly:

a) Write the eigenstates of $\hat{x} = c(a + a^\dagger)$ for a single harmonic oscillator in terms of creation operators acting on the vacuum. That is, find $g_{\omega}(q)$ such that $\hat{x} | \psi \rangle = \omega | \psi \rangle$ where $| \psi \rangle = g_{\omega}(a^\dagger) | 0 \rangle$.

b) Generalize the above construction to field theory. For a free scalar field theory, find the eigenstates $| \Phi \rangle$ of $\phi(\vec{x})$ which satisfy $\phi(\vec{x}) | \Phi \rangle = \Phi(\vec{x}) | \Phi \rangle$.

c) Write explicitly the ground-state wave functional in a free scalar theory. That is, what is $\psi_0 | \Phi \rangle = | 0 \rangle \Phi$? Express your answer in terms of the Fourier components $\phi_p = \int d^3x e^{ipx} \Phi(x)$.