Problem Set 8

1. Møller scattering is the process $e^- e^- \rightarrow e^- e^-$. Of the tree-level processes in QED this is especially interesting because it involves identical particles.
   a) Calculate the spin-averaged differential cross section for Møller scattering $e^- e^- \rightarrow e^- e^-$. Express your answer in terms of $s, t, u$ and $m_e$.
   b) Show that in the non-relativistic limit you get
   \[
   \frac{d\sigma}{d\Omega} = \frac{m_e^2 \alpha^2}{E_{cm}^2 p^2} \left( \frac{1 + 3 \cos^2 \theta}{\sin^4 \theta} \right), \quad p^2 = \left( \frac{E_{cm}}{2} \right)^2 - m_e^2 \]
   (1)
   c) Simplify the Møller scattering formula in the ultra-relativistic limit ($m_e \rightarrow 0$).
   [Hint: you should get something proportional to $(3 + \cos^2 \theta)^2$].

2. Spin vs helicity. The goal is to think about spin and helicity physically.
   a) Use the left and right chirality projection operators to show that the QED vertex $\bar{\psi} \gamma^\mu \psi \gamma^\mu \gamma^5$ vanishes unless $\psi$ and $\psi'$ have the same chirality.
   b) For the non-relativistic limit, choose explicit spinors for a spinor at rest. Show that $\bar{\psi} \gamma^\mu \psi \gamma^\mu \gamma^5$ vanishes unless $s = s'$.
   c) Use the Schrödinger equation to show that in the non-relativistic limit, the electric field cannot flip an electron’s spin, only the magnetic field can.
   d) How can you measure the spin of a slow electron?
   e) Suppose you have a radioactive source, like Cobalt-60, which undergoes $\beta$ decay $^{60}_{27}$Co $\rightarrow ^{60}_{28}$Ni $+ e^- + \bar{\nu}$. How could you (in principle) find out if those electrons coming out are polarized, that is, if they all have the same helicity? Do you think they would be polarized? If so, which polarization do you expect more of?

3. Supersymmetry
   a) Show that the Lagrangian
   \[
   \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^1 \bar{\sigma}^\mu \partial_\mu \chi + F^* F + m\phi F + \frac{i}{2} m \chi^T \sigma^2 \chi + h.c. \]  
   (2)
   is invariant under
   \[
   \delta \phi = -i\epsilon^T \sigma^2 \chi \]
   (3)
   \[
   \delta \chi = \epsilon F + \sigma^\mu \partial_\mu \phi \sigma^2 \epsilon^* \]
   (4)
   \[
   \delta F = -i\epsilon^T \sigma^\mu \partial_\mu \chi \]
   (5)
   where $\epsilon$ is an infinitesimal spinor, $\chi$ is a spinor, and $F$ and $\phi$ are scalars. All spinors anti-commute. $\sigma^2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right)$ is the second Pauli matrix.
   b) The field $F$ is an auxiliary field, since it has no kinetic term. A useful trick for dealing with auxiliary fields is to solve their equations of motion exactly and plug the result back into the Lagrangian. This is called integrating out a field. Integrate out $F$ to show that $\phi$ and $\chi$ have the same mass.
   c) *Auxiliary fields like $F$ act like Lagrange multipliers. One reason to keep the auxiliary fields in the Lagrangian is because they make symmetry transformations exact at the level of the Lagrangian. After the field has been integrated out, the symmetries are only guaranteed to hold if you use the equations of motion. Still using $\delta \phi = i\epsilon^T \sigma^2 \chi$, what is the transformation of $\chi$ that makes the Lagrangian in b) invariant, if you are allowed to use the equations of motion?
4. * SU(2) and SO(3). We discussed how SO(3) is not simply connected, as there are non-contractible paths. In this problem, we’ll explore the relation between SO(3) and its universal cover SU(2).

One way to think of SU(2) as the space of **quaternions**. Quaternions are a generalization of complex numbers. You can think of \( \mathbb{C} \) as points in the plane \( \mathbb{R}^2 \) with a norm, multiplication and division rule. Quaternions \( \mathbb{H} \) are points in \( \mathbb{R}^4 \) with a norm, multiplication and division rule. A general quaternion is \( q = a + bi + cj + dk \) with \( a, b, c, d \in \mathbb{R} \) satisfying \( i^2 = j^2 = k^2 = -1 \) and \( ij = k \). So quaternions are like complex numbers with 3 different \( i \)’s. Multiplication of quaternions is associative 

\[
q_1 (q_2 q_3) = (q_1 q_2) q_3.
\]

a) Show that \( ji = -k \) so that multiplication of quaternions is non-commutative.

b) Show that \( \bar{q} = a - bi - cj - dk \) satisfies \( \bar{q}q \in \mathbb{R} \) and that quaternions have a norm, 

\[
\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2} > 0.
\]

C) Calculate the inverse of \( q = a + bi + cj + dk \).

By calculating this, you’ve shown that quaternions are a real, normed, division algebra (in fact, \( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{H} \) are the only real normed division algebras – perhaps this is why we live in 4 dimensions!)

d) Show that \( i = i \sigma^2 \), \( j = i \sigma^1 \) and \( k = i \sigma^3 \), with \( \sigma^j \) the Pauli matrices, have all the multiplication rules of the quaternions. Thus the quaternions with \( a = 0 \) (the pure quaternions) are isomorphic to the fundamental representation of the algebra su(2).

e) Exponentiating the su(2) algebra gives the group SU(2) of unitary matrices with determinant 1. Show that \( \det(\exp(q)) = 1 \) for \( q \) a matrix in su(2) and that, for \( q \) a pure quaternion, \( \|\exp(q)\| = 1 \). Thus, the SU(2) group is quaternions of norm 1 (and so SU(2) has the topology of the surface of a 4D sphere, \( S^3 \)).

f) If we think of quaternions as vectors in \( \mathbb{R}^4 \), then pure quaternions are vectors in \( \mathbb{R}^3 \). For a pure quaternion \( p \) and a general quaternion \( q \), show that \( p' = qpq^{-1} \) is pure quaternion with \( \|p'\| = \|p\| \). Thus quaternions act on vectors as rotations.

g) To map SU(2) onto SO(3), note that we can write any SU(2) element as \( q = \exp(\theta \hat{n}) \) with \( -\pi \leq \theta < \pi \) and \( \hat{n} = n_x i + n_y j + n_z k \) and \( \hat{n} \cdot \hat{n} = 1 \). As a special case, take the quaternion \( q = \exp(k\theta) \). Show that acting on the pure quaternion \( p = \hat{x} = i \) via \( p \to qpq^{-1} \), \( q \) produces a rotation by an angle \( 2\theta \).

The general rule is \( q = \exp(\theta \hat{n}) \) rotates around \( \hat{n} \) by an angle \( 2\theta \). Thus, there are two elements of SU(2) for each rotation (the two unit norm quaternions \( q \) and \( -q \) and so SU(2) is the double cover of SO(3).

h) How could you use quaternions to describe Lorentz transformations?

A fun way to think about quaternions is as follows. Classical fields are real functions. In quantum mechanics, wavefunctions (without spin) are complex. In quantum field theory, fermions are quaternions! Quaternions are sometimes used in special relativity, and sometimes in computer graphics as a smooth and efficient way to perform rotations. Unfortunately, they have never found a “killer app” in QFT, and nobody ever uses them for practical calculations.