Pre-reading for Lecture 5a: Intro to Rotation

- A lot of new terms and variables in this lecture, but each one has an analogous concept from linear motion. So think about “translating” between linear and rotational motion.

- The simplest kind of rotation involves a rigid object (like a wheel) rotating around a fixed axis (an axle or a hinge). In this lecture we’ll focus solely on this type of rotation.

Many of the concepts from linear motion translate directly into rotational motion. Instead of distance (meters), we’ll have angle of rotation (which we’ll usually use in radians). Every linear quantity has a rotational analogue:

- Position $x$ (m) $\Leftrightarrow$ Angle $\theta$ (radians)
- Displacement $\Delta x$ (m) $\Leftrightarrow$ Angular displacement $\Delta \theta$ (rad)
- Velocity $v_x$ (m/s) $\Leftrightarrow$ Angular velocity $\omega$ (rad/s)
- Acceleration $a_x$ (m/s$^2$) $\Leftrightarrow$ Angular acceleration $\alpha$ (rad/s$^2$)
- Force $F_x$ (kg·m/s$^2$) $\Leftrightarrow$ Torque $\tau$ (kg·m$^2$/s$^2$)
- Mass $m$ (kg) $\Leftrightarrow$ Moment of inertia $I$ (kg·m$^2$)

Many of the equations for linear motion can be “translated” directly into equations for rotational motion: e.g., linear kinetic energy $K = \frac{1}{2}mv^2$ becomes rotational $K_{rot} = \frac{1}{2}I\omega^2$.

- Learning objectives: After this lecture, you will be able to...

1. Define and use angular quantities to represent position, displacement, velocity, and acceleration for an object rotating around a fixed axis.

2. Convert between radians, revolutions and degrees for rotational quantities, and use the correct dimensions with these quantities.

3. Relate the linear velocity and acceleration of a point with the angular velocity and angular acceleration (decomposed into centripetal and tangential components)

4. Use the relationships for rotational kinematics with constant angular acceleration to solve problems relating (angular) displacement, velocity, acceleration, and time.

5. Define torque and determine the torque that results from a force applied to an object (including the sign of the torque, for rotation around a fixed axis).

6. Use Newton’s Second Law for rotation to relate torque with angular acceleration

7. Define the moment of inertia of an object and calculate the moment of inertia of an object made up of a small number of “point” particles.

8. Calculate the rotational kinetic energy of an object.

9. Use energy conservation to derive the equation of motion for a simple pendulum (in the “small angle” approximation).
Defining angular quantities

- Consider a point $P$ on a wheel that is rotating counterclockwise around an axis. We'd like to describe the motion of this point, so we'll need:
  - Radius $R$ (m)
  - Angle $\theta$ (rad)
  - Angular displacement $\Delta \theta$ (rad)
  - Angular speed $\omega$ (rad/s)
  - Angular acceleration $\alpha$ (rad/s$^2$)

- We've defined these quantities in terms of radians. What other "units" could we use to measure rotational quantities? How can we convert between these different units?

<table>
<thead>
<tr>
<th>Time around circle</th>
<th>Radians</th>
<th>Degrees</th>
<th>Revolutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2\pi$rad</td>
<td>$360^\circ$</td>
<td>1</td>
</tr>
</tbody>
</table>
Activity 1: Angular vs Tangential Motion

1. A point on the rim of a wheel of radius $R$ travels one full revolution ($2\pi$ radians) around the circle. What distance has it traveled? (This is the tangential distance).

2. A point on the rim of a wheel of radius $R$ starts out at angle $\theta_1$ and ends at angle $\theta_2$. What distance has it traveled (along the circumference)? (Hint: use radians for your angles.)

3. A wheel of radius $R$ is rotating at a constant angular velocity $\omega$ (radians per second). What is the tangential velocity (magnitude) of a point on the rim? (your answer for velocity should be in terms of $\omega$ and $R$)

$$V_T = \frac{\text{dist}}{\text{time}} = \frac{\Delta \theta R}{\Delta t} = \omega R$$

4. What is the acceleration (magnitude) of a point on the rim of a wheel of radius $R$ rotating at a constant angular velocity $\omega$? (your answer should be in terms of $\omega$ and $R$) Also, describe the direction acceleration points, in words. Hint: if you already know an expression for the centripetal acceleration of something moving in a circle... start with that.

$$a = \frac{V^2}{R} = \frac{(\omega R)^2}{R} = \omega^2 R$$

5. Now suppose that the angular velocity is changing, so now there’s an angular acceleration $\alpha$ (rad/s$^2$). What is the tangential acceleration? (your answer should be in terms of $\alpha$ and $R$) Hint: start with the definition of acceleration.

$$a_T = \frac{dv}{dt} \text{ or } a_T = \frac{dV_T}{dt} = \frac{d(\omega R)}{dt} = R \frac{d\omega}{dt} = R \alpha$$

• **Bonus!** Find the components of the (vector) acceleration $\vec{a}$ if the angular velocity is not constant. (assume the positive $y$-direction points from the axis of rotation to the point P.)

$$\vec{a} = (-a_{\text{tan}}, -a_R) = (-R \alpha, -\omega^2 R)$$
Activity 2: Rotational kinematics with constant angular acceleration

1. Make the following changes to both sides of the 1-dimensional kinematic equations:

   **Velocity vs. time:**   \[ v = v_0 + at \]  
   Divide by \( R \):
   \[ \frac{v}{R} = \frac{v_0}{R} + \frac{a}{R} \cdot t \]  
   \[ \frac{v}{R} - \frac{v_0}{R} = \frac{a}{R} \cdot t \]  
   \[ \frac{v}{R} = \frac{a}{R} \cdot t + \frac{v_0}{R} \]  

   **Position vs. time:**   \[ x = x_0 + v_0 t + \frac{1}{2} a t^2 \]  
   Divide by \( R \):
   \[ \frac{x}{R} = \frac{x_0}{R} + \frac{v_0}{R} t + \frac{1}{2} \frac{a}{R} t^2 \]  
   \[ \frac{x}{R} = \frac{x_0}{R} + \frac{v_0}{R} t + \frac{1}{2} \frac{a}{R} t^2 \]  

   **Velocity vs. position:**   \[ v^2 - v_0^2 = 2a(x - x_0) \]  
   Divide by \( R^2 \):
   \[ \frac{v^2}{R^2} - \frac{v_0^2}{R^2} = \frac{2a}{R} \left( \frac{x - x_0}{R} \right) \]  
   \[ \frac{v}{R} \cdot \frac{v_0}{R} = \frac{a}{R} \left( \frac{x}{R} - \frac{x_0}{R} \right) \]  

2. Now "translate" these three equations into analogous equations for rotational motion with constant angular acceleration. (Hint: can you translate from Latin to Greek?)

   **Vel. vs. time:**   \[ \omega = \omega_0 + \alpha \cdot t \]  
   **Pos. vs. time:**   \[ \theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2} \alpha \cdot t^2 \]  
   **Vel. vs. pos.**   \[ \omega^2 - \omega_0^2 = 2\alpha (\theta - \theta_0) \]

3. A CD has a maximum speed of about 1,050 rad/s. Its radius is 6 cm. Starting from rest, the disk reaches its maximum speed in 3 seconds. What is the angular acceleration required? (Assume it is constant.)

   \[ \theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2} \alpha \cdot t^2 \]  
   \[ \omega = \omega_0 + \alpha \cdot t \]  
   \[ \omega = 1050 \text{ rad/s} \]  
   \[ t = 3 \text{ s} \]  
   \[ \alpha = \frac{\omega}{t} = 350 \text{ rad/s}^2 \]

- **Bonus!** How many times does the disk revolve while it is spinning up?

   \[ \theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2} \alpha \cdot t^2 \]  
   \[ \theta = 1575 \text{ rad} \]  
   \[ \theta = \frac{350}{25} \text{ rev} \]
Defining Torque

- You want to open the door, so you push on the door. Where should you push? (The circle represents the door hinge.)

- If the door somehow resists turning, your ability to open the door will depend on:
  - the magnitude of the force
  - the direction of the force
  - the point of application of the force

These quantities combine to yield the torque. Torque is the rotational analogue of force:

\[ \tau = RF \sin \theta \]

- We'll use a sign convention for the torque:
  \[ \tau \text{ is } \theta \text{ counterclockwise} \]

- What are the units of torque?
  \[ \text{N} \cdot \text{m} \] (not joules)

- Other ways of thinking about torque:

\[ \tau = RF \sin \theta = R F_\perp \]

\[ \tau = FR \sin \theta = FR_\perp \]

\[ R_\perp = h \frac{F}{R} \]

\[ \tau = Fh \]
Activity 3: Rotational analogues of Force, Mass, and Kinetic Energy

1. In one dimension, Newton’s Second Law reads: $\sum F_x = ma_x$.
   - What is the rotational analogue of $a_x$? $\tau$
   - What is the rotational analogue of $F_x$? $I$
   - The rotational analogue of mass is called the moment of inertia, $I$. Write the rotational analogue of Newton’s Second Law:

   \[ \Sigma \tau = I \alpha \]

2. A small mass $m$ is revolving around an axis at a radius $R$. You apply a tangential force $F_{\text{tan}}$ to the mass as shown in the figure.
   - Write the tangential component for Newton’s Second Law (this would be the $y$-component in the figure):

   \[ F = ma \]

   - Write the rotational version of Newton’s Second Law. (Hint: what is the torque produced by that force?)

   \[ \tau = I \alpha \]

   - Put these two equations together to find the moment of inertia $I$ for a point mass $m$ revolving at a radius $R$. (You’ll need the definition of tangential acceleration $a_{\text{tan}}$. Ask a TF if your team is stuck here.)

   \[ I = \frac{\tau}{\alpha} = \frac{F R \sin \theta}{a_{\text{tan}} / R} = \frac{F R^2}{a_{\text{tan}}} = m R^2 \]

3. If the mass $m$ is revolving at a constant speed $v$, what is its kinetic energy?

   \[ \frac{1}{2} m v^2 \]

   Can you find an equivalent expression for kinetic energy in terms of rotational quantities?

   \[ \frac{1}{2} I \omega^2 \]

   • Bonus! Find the moment of inertia of the (asymmetric) dumbbell shown at right if it rotates around its own center of mass.

   \[ I = \sum m_i R_i^2 \]
Physical Sciences 2: Lecture 5a

One-Minute Paper

Your name: _______________________________ TF: _______________________________

- Please tell us any questions that came up for you today during lecture. Write “nothing” if no questions(s) came up for you between 6–9pm (or while viewing it online).

- What single topic left you most confused after today’s class?

- Any other comments or reflections on today’s class?