Problem Set 5

1. Radioactive decay. The muon decays to an electron and two neutrinos through an intermediate massive particle called the $W^-$ boson. The muon, electron and $W^-$ all have charge $-1$.

   a) Write down a Lagrangian that would allow for $\mu^- \to W^- \nu_\mu \to e^- \bar{\nu}_e \nu_\mu$. Assume the $W$ and other particles are all scalars, and the $e^-, \nu_e$ and $\nu_\mu$ are massless. Call the coupling $g$. (NB: this is not actually correct, because the particles are not scalars, but the result will only differ by a factor of order 1 from our computations here.)

   b) Calculate $|M|^2$ for this decay in the limit that the $W$ mass, $m_W$, is large.

   c) The decay rate $\Gamma = \frac{1}{\text{lifetime}}$ is proportional to $|M|^2$. The real weak-interaction coupling is dimensionless, like the coupling $e$ for the photon, but your coupling $g$ is dimensionful. This difference is because we ignore spin. If the $W$ spin were included, you would get extra factors of $p^\mu$, which would turn into factors of $\sqrt{2} = m_\mu$ in $|M|^2$. Use dimensional analysis to figure out what power of $m_\mu$ should be there for $g$ to be dimensionless. Also, throw in a $\frac{1}{192\pi^3}$ for the 3-body phase space, as you computed in problem set 3, to get an approximate formula for $\Gamma$.

   d) Look up the muon mass and lifetime on pdglive.lbl.gov. Let’s guess that the coupling $g$ is close to the coupling for electromagnetism, so $\frac{g^2}{4\pi} \approx \frac{1}{137}$. Use this and the muon mass and lifetime to estimate the $W$ mass.

   e) The tauon, $\tau$, also decays to $e^- \bar{\nu}_e \nu_\tau$. Look up the $\tau$ lifetime on pdglive and use it and previous parts to estimate the $\tau$ mass. Which of $m_W, g, m_\mu$ the muon lifetime, or the $2\pi$’s we threw in does your prediction depend on?

   f) Look up the branching ratio $\tau \to e^- \bar{\nu}_e \nu_\tau$, listed on pdglive as a Fraction($\tau_e$). This is the fraction of time $\tau$ decays this way. Use this to refine your tau mass estimate.

   g) How could you measure $g$ and $M_W$ separately using very precise measurements of the $\mu$ and $\tau$ decay distributions? What precision would you need (in %)?

2. Unstable particles. Unstable particles pick up imaginary parts which generate a width $\Gamma$ in their resonance line shape. This problem will develop an understanding of what is meant by the terms width and pick up.

   a) What would the cross section be for s-channel scattering if the intermediate propagator were $\frac{i}{p^2 - m^2 + i\Gamma}$, where $\Gamma > 0$? This is called the Breit-Wigner distribution.

   b) Sketch the cross section as a function of $x = \frac{s}{m^2}$ for $\frac{\Gamma}{m}$ small and for $\frac{\Gamma}{m}$ large.

   c) Show that a propagator only has an imaginary part if it goes on-shell. Explicitly, show that $\text{Im}(\mathcal{M}) = -\pi\delta(p^2 - m^2)$, when $\Gamma = \frac{i}{p^2 - m^2 + i\epsilon}$.

   d) Loops of particles can produce effective interactions which have imaginary parts. Suppose we have another particle $\psi$ and an interaction $\phi \bar{\psi} \psi$ in the Lagrangian. Loops of $\phi$ will have imaginary parts if and only if $\phi$ is lighter than half of $\phi$, that is, if $\phi \to \psi \bar{\psi}$ is allowed kinematically. Draw a series of loop corrections to the $\phi$ propagator. Show that if these give an imaginary number, you can sum the graphs to reproduce the propagator in part a).

   e) What is the connection between parts c and d? Can you see why the width is related to the decay rate?

3. Calculate the classical propagator for a massive spin 1 particle by inverting the equations of motion to the form $A_\mu = \Pi_{\mu\nu} L_\nu$. 
4. Vector polarization sums. In this problem we build some intuition for the way in which the numerator of a spin 1 particle propagator represents an outer product of physical polarizations $|\epsilon\rangle\langle\epsilon|$. Calculate the $4 \times 4$ matrix outer product $|\epsilon\rangle\langle\epsilon|\equiv \sum_i \epsilon_i^j \epsilon^i_j$ by

a) summing over the physical polarizations for a massive spin 1 particle in some frame. Re-express your answer in a Lorentz covariant way, in terms of $m$, $k_\mu k_\nu$ and $g_{\mu\nu}$.

b) show that the numerator of the massive vector propagator (problem 3) is the same as the polarization sum. Why should this be true?.

c) summing over the two physical polarizations for massless vector. A helpful basis for these polarizations is choosing them to be orthogonal to momenta, $\epsilon^i \cdot p = 0$ and to an arbitrary reference vector $r^\mu$: $\epsilon^i \cdot r = 0$. Find explicit forms for the two polarizations, do the sum, and then express your answer in a Lorentz covariant way (i.e. in terms of $p^\mu$ and $r^\mu$).

d) write down a Lagrangian so that the photon propagator derived from it has the numerator you found in part c).

e) * Compare the numerator from part (c) to the numerator of the photon propagator in the $R\xi$ gauges. What might be an advantage of using the numerator from (c) rather than Feynman gauge? What might be a disadvantage?

5. * Tensor polarization sums. A spin 2 field can be embedded in a 2-index tensor $h_{\mu\nu}$. Therefore the polarizations are tensors too, $\epsilon^i_{\mu\nu}$. These should be orthonormal, $\epsilon^i \cdot \epsilon^j \equiv g^{\mu\nu}g^{\alpha\beta}\epsilon^i_{\mu\alpha}\epsilon^j_{\nu\beta} = \delta^{ij}$,

a) The polarizations should be transverse, $k_\mu \epsilon^i_{\mu\nu} = 0$ and symmetric $\epsilon^i_{\mu\nu} = \epsilon^i_{\nu\mu}$. How many degrees of freedom do these conditions remove ($\epsilon_{\mu\nu}$ started with 16 dof)?

b) For a massive spin 2 field, we can work in the rest frame. In this frame, how many orthonormal $\epsilon^i_{\mu\nu}$ can you find? Write your basis out explicitly, as $4 \times 4$ matrices.

c) Guess which of these correspond to spin 0, spin 1, or spin 2. What kind of Lorentz-invariant condition can you impose so that you just get the spin 2 polarizations?

d) If you use the same conditions but take $k_\mu$ to be the momentum of a massless particle, what are the polarizations? Do you get the right number?

e) What would you embed a massive spin 3 field in? What conditions could you impose to get the right number of degrees of freedom?

6. * Show that is impossible to write down a Lorentz-invariant Lagrangian for a single scalar field with 4-derivative kinetic terms (e.g. $\mathcal{L} = \phi \Box^2 \phi$) that generates non-negative energy density.