PART A (Graded by Jack)

PROBLEM 1 (3+2 points, suggested length of 1/4 page)

(A) Prove that in every undirected graph, there are an even number of vertices of odd degree. *Hint:* Use the Handshaking Lemma presented in the Undirected Graphs mini-lecture.
(B) Prove that at a party where some people shake hands, the number of people who shake hands with an odd number of people is an even number.

Solution.
PROBLEM 2 (2+1+2 points, suggested length of 1/2 page)

Why shouldn’t Jack have an algorithm named after him just like Kruskal and Prim? Jack designs “Kelley’s Algorithm” for constructing a minimum-weight spanning tree from a connected, weighted simple graph $G$, that works as follows:

$$M = \{V, E\} := G$$

sort $E$ from maximum weight to minimum weight
for each $e \in E$
  remove $e$ only if removing it would not disconnect $M$
return $M$

Consider the following graph, $G$, with edge weights as follows:

(A) Construct a minimum-weight spanning tree for $G$ using Kelley’s Algorithm.

(B) What is the total weight of the spanning tree generated by Kelley’s Algorithm?

(C) Prove that Kelley’s Algorithm is correct.

Solution.
PART B (Graded by Theresa)

PROBLEM 3 (1+1+1 points, suggested length of 1 page)

The following is a map of the western United States.

(Note: CO is not adjacent to AZ and UT is not adjacent to NM)
(A) Draw a graph that represents the adjacency of states. (There should be a vertex for each state and an edge should connect two states if and only if the two states are adjacent.)
(B) Find a four coloring of the graph.
(C) Find a three coloring, or explain why no three coloring can exist.

Solution.
PROBLEM 4 (3 points, suggested length of 1/4 page)

Prove that if a connected graph $G$ has no cycles, then it has a unique spanning tree.

Solution.
Below is a graph of some cities and the roads and ferries connecting them showing the maximum number of inches of snow that can fall before travel by that route becomes impossible. For example, if 10 inches of snow fall, the direct Plymouth to Onset route is closed, but one can get from Plymouth to Onset via Taunton. Snowfall is always uniform everywhere across the region.

(A) Ignoring the labels, what is the edge connectivity of the region? What is the vertex connectivity? Justify your answers.

(B) Now following the snowfall criterion for edge removal, what is the maximum amount of snowfall that would leave the region connected? Justify your answer.

(C) Does the graph have any articulation points? Does it have any bridges? How would your answers change if 5 inches of snow fell on the region? Identify the articulation points or bridges if appropriate.

Solution.