PART A (Graded by Ahmed)

PROBLEM 1 (2+2+2 points, suggested length of 1/2 page)

(A) Use diagonalization to prove that the set of real numbers in the interval (0, 1), that is \( \{ r \in \mathbb{R} : r > 0 \land r < 1 \} \), is uncountably infinite.

(B) The Schröder-Bernstein Theorem states that for sets \( S \) and \( T \), if there exist injective functions \( f : S \rightarrow T \) and \( g : T \rightarrow S \), then \( S \) and \( T \) have the same cardinality. Either by using the Schröder-Bernstein Theorem or by defining a bijection, show that the cardinality of the set of real numbers in the closed interval \([0, 1]\) is the same as the cardinality of the set of all real numbers in the open interval \((0, 1)\).

(C) Prove that there are at least a countably infinite number of uncountably infinite sets.

Solution.

PROBLEM 2 (2+2+2 points, suggested length of 1/3 page each)

A robot named Wall-E wanders around a two-dimensional grid. He starts at \((0, 0)\) and is allowed to take four different kinds of steps:

1. \((+2, 0)\)
2. \((-2, -2)\)
3. \((+1, +2)\)
4. \((-3, 0)\)

For example, Wall-E might take the following stroll:

\[
(0,0) \rightarrow^1 (2,0) \rightarrow^2 (3,2) \rightarrow^4 (0,2) \rightarrow^2 (-2,0) \rightarrow \ldots
\]

(A) Describe a state machine model of this problem. How can a state be represented? What is the start state? Given a state, what are the transitions out of it?

(B) Wall-E’s true love, the fashionable and high-powered robot Eve, awaits at \((10, 13)\). Can Wall-E
ever find his true love? Either find a path from Wall-E to Eve or prove that Eve’s location cannot be reached by demonstrating that it does not meet a preserved invariant.

(C) If Eve was instead waiting at (27, 16), could Wall-E ever find her? Either find a path from Wall-E to Eve or prove that Eve’s location cannot be reached by demonstrating that it does not meet a preserved invariant.

Solution.

**PART B (Graded by Jack)**

**PROBLEM 3** (4 points, suggested length of 1/2 page)

Let \( m, n \in \mathbb{Z} \) where \( m \neq 0 \) and \( n \neq 0 \). Define a set of integers \( L \) as follows:

- **Base cases:** \( m, n \in L \)
- **Constructor cases:** If \( j, k \in L \), then
  1. \(-j \in L\)
  2. \(j + k \in L\)

Prove by structural induction that every common divisor of \( m \) and \( n \) also divides every member of \( L \).

Solution.

**PROBLEM 4** (2+2 points, suggested length of 1/2 page)

Draw state machines that only accept strings in the following sets. Assume that the alphabet is \( \Sigma = \{0, 1\} \), that is that all strings \( s \in \Sigma^* \).

(A) \( \{w : w \text{ contains the substring 0110, i.e. } w = x0110y \text{ for some strings } x, y \in \Sigma^* \}\)

(B) \( \{w : w \text{ has an even number of 0s and an odd number of 1s} \}\)

Solution.
Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE**
(See the syllabus for information)**