PART A (Graded by Allison)

PROBLEM 1 (2+2 points, suggested length of 1/4 page each)

In this problem, we will prove that for any two sets $X$ and $Y$, $X \cap Y = X \cup Y$. We will do this by proving $X \cap Y \subseteq X \cup Y$ in part (A) and $X \cup Y \subseteq X \cap Y$ in part (B). If two sets are subsets of each another, then they must be equal.

(A) Let $a \in X \cap Y$. Prove $a \in X \cup Y$.
(B) Let $b \in X \cup Y$. Prove $b \in X \cap Y$.

Solution.

PROBLEM 2 (4 points, suggested length of 2/3 page)

Prove by induction that, given a finite set $S$, $|P(S)| = 2^{|S|}$.

Solution.

PROBLEM 3 (4 points, suggested length of 1/2 page)

Find a bijection between the set of all multiples of $\frac{1}{2}$ and the set of all natural numbers (including 0). Be sure to explain why your relation is bijective.

Solution.

PART B (Graded by Rebecca)

PROBLEM 4 (1+2+0 points, suggested length of 1/2 page each)
(A) Prove that \(|A \cup B| = |A| + |B| - |A \cap B|\).
(B) Using your answer to part A, derive a formula for the cardinality of the union of three sets: \(|A \cup B \cup C|\). Your final formula should not contain any unions.
(C) (OPTIONAL) Give a formula (in mathematical notation or English) for the cardinality of a union of \(n\) sets:
\[
\left| \bigcup_{i=1}^{n} S_i \right|
\]

Solution.

PROBLEM 5 (5 points, suggested length of a few sentences each)

For each of the following functions, decide whether it is injective, surjective, and/or bijective. If the function is a bijection, what is its inverse? If it is injective but not surjective, what is its inverse on the image of its domain?

(A) \(f : \mathbb{Z} \to \mathbb{Z}, \) where \(f(n) = 2n\).
(B) \(f : \mathbb{R} \to \mathbb{Z}, \) where \(f(x) = \lfloor x \rfloor\).
(C) \(f : \mathbb{R} \to \mathbb{R}, \) where \(f(x) = \lfloor x \rfloor\).
(D) \(f : \mathbb{Z} \to \mathbb{R}, \) where \(f(n) = \frac{n}{2}\).
(E) \(f : \mathbb{R} \to \mathbb{R}, \) where \(f(x) = \frac{x}{3}\).

Solution.
Problem set by **FILL IN YOUR NAME HERE**

Collaboration Statement: **FILL IN YOUR COLLABORATION STATEMENT HERE**
(See the syllabus for information)**