Physical Sciences 2: Assignments for Nov 7 – 14, 2017
Homework #9: High and Low Reynolds Number Flow, Statistical Mechanics
Due Tuesday, Nov. 21, at 9:30AM

After completing this homework, you should…

- Be able to quantitatively and qualitatively describe the meaning of the Reynolds number
- Understand the differences between high Re flow and low Re flow
- Be able to explain the concept of dynamical similarity
- Understand the derivation of the continuity equation
- Know the derivation of and how to use Bernoulli’s equation
- Be able to qualitatively and quantitatively explain the meaning of viscosity
- Be able to use Newton’s law of viscosity
- Understand the properties of low Re number flows
- Be able to use the Stokes equation
- Be able to use the Poiseuille equation
- Know the meaning of Brownian motion
- Know how statistical mechanics is used to describe pressure and temperature
- Be able to calculate the average kinetic energy and average velocity of gas molecules
Reynolds Number

- The Reynolds number $RE$ is a number used to
  - determine whether a flow is laminar ($RE < 2000$) or turbulent ($RE > 2000$)
  - determine whether viscous drag ($RE << 1$) or pressure drag ($RE >> 1$) dominates
    \[ F_{\text{drag}} = \frac{1}{2} C_d A R \nu \]
  - determine the characteristics of different types of flow (high $RE$ flow vs. low $RE$ flow)

\[ RE = \frac{\nu l}{\eta} \]

High $RE$ Flow

- $RE = \frac{\nu l}{\eta}$ large; could be a large object moving at a normal speed in a fluid of normal viscosity.

- Bernoulli Equation
  - relates pressure, height, and velocity at two points in a high Reynolds flow
  \[ P_1 + sgh_1 + \frac{1}{2} s \nu_1^2 = P_2 + sgh_2 + \frac{1}{2} s \nu_2^2 \]

- Venturi Effect
  - relates pressure and velocity in a high Reynolds flow
  \[ P_1 - P_2 = \frac{1}{2} s (\nu_2^2 - \nu_1^2) \]
  - lower pressure, higher velocity in the narrow portion of tube

\[ \text{Diagram of venturi effect} \]
Low RE Flow

- $RE = \frac{\text{Re}}{\eta}$ small, could be small object moving slowly through a fluid with any viscosity, could be a large object moving at a low speed through a fluid with a high viscosity.

Stoke's Equation

- Drag force acting on an object moving through a low RE number fluid

\[ F_{\text{drag}} = 6\pi \eta VR \]

- Poiseuille Equation

- Relate the flow rate to the pressure change and viscosity

\[ Q = \frac{\pi}{8} \frac{\Delta P}{\eta^2} R^4 \]

Statistical Mechanics

- Cannot track every single molecule; instead only consider averages

- The average kinetic energy of a molecule is

\[ \langle KE \rangle = \frac{3}{2} k_B T \]

- The RMS (root-mean-square) velocity of a molecule is

\[ \langle v^2 \rangle = \frac{3}{2} k_B T \]

\[ V_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{2\langle KE \rangle}{m}} \]

- This is the speed of a molecule that has the average kinetic energy; $V_{\text{rms}}$ is very close to the average speed of a typical molecule.
1. Squeeze-It! (1 pt)

The schematic at right shows a squirt bottle of the sort demonstrated in class (and used in many chemistry labs). When you squeeze the body of the bottle, you increase the pressure inside the bottle slightly. Given the approximate pressures and dimensions shown, what is the speed of the water as it emerges from the nozzle? (You may assume that the cross-sectional area of the bottle is much larger than the area of the nozzle.)

Since we are dealing with water—a not very viscous fluid, and the velocity of the exiting fluid $v$ is of order 1 m/s, we can assume the Reynold’s number is large enough to use Bernoulli’s equation to describe the motion of the fluid.

$$P_i - P_f + \rho g (h_i - h_f) = \frac{1}{2} \rho (v_f^2 - v_i^2)$$

Another key point to this problem is choosing your initial pressure to be at the top of the fluid inside the jar, $P_i = 1.1 \times 10^5$ Pa, which is approximately not moving, $v_i = 0$, since the width of the jar is large compared to the width of the nozzle. If we choose the initial pressure right at the bottom of the nozzle, we run into the problem of not knowing the initial velocity of the water right at the bottom of the straw.

Setting the initial height $h_i = 0$ and final height to $h_f = 0.10$ m, we can find the final velocity:

$$v_f = \sqrt{\frac{2 (P_i - P_f + \rho g (h_i - h_f))}{\rho}}$$

$$v_f = \sqrt{\frac{2 \left( (1.1 - 1) \times 10^5 \text{Pa} + 1000 \frac{\text{kg}}{\text{m}^3} \left( \frac{9.8 \text{m}}{\text{s}^2} \right) (0 - 0.1 \text{m}) \right)}{1000 \frac{\text{kg}}{\text{m}^3}}}$$

$$= \sqrt{\frac{2 \left( 0.1 \times 10^5 \frac{\text{kg}}{\text{m} \cdot \text{s}^2} - 100 \frac{\text{kg}}{\text{m}^2} \left( \frac{9.8 \text{m}}{\text{s}^2} \right) \right)}{1000 \frac{\text{kg}}{\text{m}^2}}} = 4.247 \text{ m/s}$$

In case we wanted to double check the Reynold’s number for this problem, we can estimate the nozzle diameter to be 1 mm find:

$$Re = \frac{\rho v D}{\eta} = \frac{(1 \times 10^3 \text{kg} / \text{m}^3)(4.25 \text{ m/s})(0.001 \text{m})}{8.94 \times 10^{-4} \text{Pa} \cdot \text{s}} \sim 4772$$
2. Clogging your artery (2 pts)

Suppose that blood flows through a large, unobstructed artery with a speed of 0.5 m/s. Assume the artery is horizontal.

a) If part of the artery becomes obstructed, and the obstructed region has a radius 1/3 that of the unobstructed region, by how much does the blood pressure drop in the obstructed region?

If the radius of the artery in the obstructed region is constricted to 1/3 of its value elsewhere, then the area is 1/9 the area of the unobstructed region. From the continuity equation, the speed therefore goes up by a factor of 9. So the speed of blood flow in the obstructed region will be 4.5 m/s.

Now we can apply Bernoulli’s principle to determine how much the pressure drops in the obstructed region. Applying Bernoulli to a streamline in the obstructed artery (where 1 labels the unobstructed region and 2 labels the obstructed region, as in the Venturi diagram above), we get:

\[ P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2 \]

\[ P_1 - P_2 = \frac{1}{2} \rho \left( v_2^2 - v_1^2 \right) \]

\[ = \frac{1}{2} \left( 1060 \text{ kg/m}^3 \right) \left( (4.5 \text{ m/s})^2 - (0.5 \text{ m/s})^2 \right) \]

\[ = 10.6 \text{ kPa.} \]

This is about 80 torr.

b) Under what conditions could that drop in pressure cause the artery to collapse?

If the gauge pressure in region 1 is less than 80 (torr), then the pressure in region 2 will be lower than atmospheric. That means the pressure on the outside of the walls of the artery will exceed the pressure on the inside. Depending upon the elasticity of the arterial walls, this could cause the artery to elastically deform or completely collapse in region 2.

We know that the blood pressure in the artery varies between the systolic and diastolic pressures. If the diastolic pressure is lower than 80, the artery will deform during each diastole and open during each systole. That’s bad news. If the systolic pressure is also less than 80, the artery will remain deformed or completely collapsed! That’s very bad news.
3. Punctured aorta! (1 pt)

During open-heart surgery, a surgeon accidentally tears a small circular hole (2 mm diameter) in the aorta. If that blood squirts straight up into the air, how high will it squirt? (Neglect air resistance.)

The speed of blood flow in the aorta is about 0.33 m/s. To determine the speed of blood squirting out the hole, we can’t use the continuity equation, because not all of the blood from the aorta comes out of the hole. However, we can apply the Bernoulli equation to a streamline that starts in the aorta and then comes out the hole.

Let’s label the flow 1 while it is still in the aorta (at a pressure of roughly 100 torr above atmospheric), and with 2 when it comes out the hole (at atmospheric pressure). Then Bernoulli’s equation gives

\[ P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2 \]

\[ v_2 = \sqrt{v_1^2 + \frac{2}{\rho} (P_1 - P_2)} \]

\[ = \sqrt{(0.33 \text{ m/s})^2 + \frac{2}{1060 \text{ kg/m}^3} (100 \text{ torr})} \]

\[ = \sqrt{0.11 \text{ m}^2/\text{s}^2 + (25 \text{ m}^2/\text{s}^2)} \]

The first term under the square root is negligible compared with the second term, so it would not have mattered if we had assumed the blood in the aorta was static; the speed that it squirts out is essentially entirely due to the pressure difference between the inside and outside of the aorta. Also note that the speed doesn’t seem to depend on the size of the hole, although the volumetric flow rate out the hole (which tells you the rate of blood loss) certainly would depend on its size. Putting in the numbers gives

\[ v_2 = 5.0 \text{ m/s}. \]

You could do this problem using ordinary kinematics, but we can also apply Bernoulli’s principle again, since the same streamline continues up to the top of the blood’s trajectory. Let’s label the flow with 3 when it reaches the top of its trajectory in the open air. Now the velocity is 0 and the pressure is still atmospheric. Then Bernoulli’s equation gives

\[ P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 = P_3 + \frac{1}{2} \rho v_3^2 + \rho g y_3 \]

\[ y_3 - y_2 = \frac{v_2^2}{2g} = \frac{(5.0 \text{ m/s})^2}{2(9.8 \text{ m/s}^2)} \]

\[ \frac{P_2 = P_3, \text{ and } v_3 = 0, \text{ so those terms dropped out of the equation. (Note that we could just as easily have applied Bernoulli’s equation to regions 1 and 3.) Calculating the numbers gives a height of}} \]

\[ y_3 - y_1 = 1.3 \text{ m}. \]
4. The giraffe, again (1 pt)

A giraffe’s heart must provide sufficient diastolic pressure to overcome both the hydrostatic pressure that results from the height of its head above its heart and the viscous resistance to fluid flow in the artery that travels from its head to its heart. Given the following information:

• Blood has a density of 1060 kg/m^3
• A giraffe’s head is about 3 meters above its heart
• The artery from the giraffe’s heart to its head has a radius of about 3 mm
• The giraffe needs to supply 9 mL of blood per second to its brain
• The viscosity of whole blood is about \(4 \times 10^{-3}\) kg·m\(^{-1}\)·s\(^{-1}\)

Calculate the diastolic pressure that must be produced by the giraffe’s heart in order for the pressure in its head to be 1 atmosphere.

This problem is similar to the derivation of the Poiseuille equation, except that this time we are dealing with a vertical element of fluid instead of a horizontal one. Consider a column of fluid of height \(h\) in a pipe of radius \(R\) that supports a steady upward flow \(Q\), as in the diagram. The forces on the column are due to pressure from the fluid above (pushing down) and below (pushing up), gravity (pulling down), and viscous shear from the walls of the pipe (pulling down). The flow is steady, so there is no change in velocity, which means the sum of the forces must be zero. The net force in the \(y\)-direction is

\[
\sum F_y = P_{\text{bottom}} \left( \pi R^2 \right) - P_{\text{top}} \left( \pi R^2 \right) - mg - \tau \left( \text{lateral area} \right)
\]

\[
= P_{\text{bottom}} \left( \pi R^2 \right) - P_{\text{top}} \left( \pi R^2 \right) - \left( \rho \pi R^2 h \right) g - \tau \left( 2\pi Rh \right).
\]

We set this equal to zero to enforce fluid flow at constant velocity and divide through by \(\pi R^2\) gives

\[
0 = P_{\text{bottom}} - P_{\text{top}} - \rho gh - \tau \left( \frac{2h}{R} \right)
\]

As we did in Lecture 9b, we can identify the shear stress \(\tau\) as

\[
\tau = \alpha \eta \frac{V}{R} = \alpha \eta \frac{Q}{\pi R^2} = \frac{\alpha \eta Q}{\pi R^3}
\]

from Newton’s Law of viscosity and \(\alpha\) is a “constant”, which comes from solving the exact geometry. In this case \(\alpha=8\) to describe fluid flow through pipes. So the pressure difference between the top and bottom of the column required to maintain a steady flow is

\[
\Delta P = P_{\text{bottom}} - P_{\text{top}} = \rho gh + \left( \frac{8\eta Q h}{\pi R^4} \right) = \Delta P_g + \Delta P_{\text{Poiseuille}}
\]
The last term is the pressure difference needed to overcome viscosity, and it’s exactly the same as given by the Poiseuille. For the giraffe, the flow rate is

\[
Q = \left( \frac{9 \text{ mL}}{s} \right) \left( \frac{10^{-6} \text{ m}^3}{\text{mL}} \right) = 9 \times 10^{-6} \text{ m}^3 / \text{s}.
\]

So the Poiseuille term (corresponding to the pressure difference needed to overcome viscosity) is

\[
\Delta P_{\text{Poiseuille}} = \frac{8\eta h Q}{\pi R^4} = \frac{8 \left( 4 \times 10^{-3} \text{ Pa} \cdot \text{s} \right) \left( 3 \text{ m} \right) \left( 9 \times 10^{-6} \text{ m}^3 / \text{s} \right)}{\pi \left( 3 \times 10^{-3} \text{ m} \right)^4} = 3.4 \text{ kPa}.
\]

The total pressure difference is the Poiseuille pressure plus the static pressure difference

\[
\rho gh = (1060 \text{ kg/m}^3) \left( 9.8 \text{ m/s}^2 \right) \left( 3 \text{ m} \right) = 31 \text{ kPa}.
\]

So the total pressure difference needed between the giraffe’s heart and head is the sum of these two:

\[
\Delta P_{\text{total}} = \Delta P_{\text{Poiseuille}} + \rho gh = 31 \text{ kPa} + 3.4 \text{ kPa} = 35 \text{ kPa}.
\]

Translated into the usual units for blood pressure, this corresponds to a diastolic pressure at the giraffe’s heart 260 torr higher than its head. Remember, blood pressures are reported as gauge pressures (pressures above atmospheric), and we want the pressure at the head to be 1 atmosphere, so this means the giraffe’s diastolic pressure at its heart exceeds atmospheric pressure by 260 torr.

In HW7, when we considered the blood pressure of the giraffe, we considered only static pressure, and arrived at an estimate of around 220 torr needed for blood to get to the giraffe’s head. Adding in the effects of viscosity increases this by around 40 torr (actually closer to 30, because we used 1000 kg/m\(^3\) for the density of blood back in HW7 instead of 1060 kg/m\(^3\)). Note, however, how sensitively this depends on the radius of the artery: using a 4-mm radius instead of 3 mm would reduce the effects of viscosity by a more than factor of 3 because of the \(R^4\) in the denominator of Poiseuille’s equation.
5. Turbulence in the bloodstream (2 pts)

For steady flow in a circular pipe, the transition between laminar and turbulent flow occurs at a Reynolds number on the order of $10^3$.

a) Estimate the Reynolds number for blood flow in the aorta and determine if the flow there is laminar, turbulent, or too close to call.

For flow in a pipe, we use the pipe’s radius (or diameter) as the length scale for purposes of calculating the Reynolds number. The radius of the aorta and speed was given in problem 3,

$$Re = \frac{\rho v R}{\eta} = \frac{(1.06 \times 10^3 \text{ kg/m}^3)(0.33 \text{ m/s})(0.9 \text{ cm})}{4 \times 10^{-3} \text{ Pa} \cdot \text{s}} = 800$$

This is apparently below the cutoff for turbulence, but our estimate isn’t really precise enough to distinguish laminar from turbulent. For example, if we had used the diameter instead of the radius, we would have gotten a number greater than $10^3$. So this one is too close to call. Indeed, it turns out that flow in your aorta is sometimes laminar and sometimes turbulent, and doctors can actually hear the difference using a stethoscope.

b) Do the same for flow in both veins and capillaries. (Hint: you might need to make some estimates about size and flow rate.)

Veins are much smaller than arteries; looking at the back of my hand, I’d estimate that a vein might be only about 1 mm in diameter. As for the speed, I found a paper from 2011 that measures blood velocity profiles in retinal vessels [Investigative Ophthalmology & Visual Science, June 2011, Vol 52, No 7, p 4151], that gives a maximum speed in veins of about 15 mm/s. That would give a Reynolds number of

$$Re = \frac{\rho v R}{\eta} = \frac{(1.06 \times 10^3 \text{ kg/m}^3)(0.015 \text{ m/s})(5 \times 10^{-4} \text{ m})}{4 \times 10^{-3} \text{ Pa} \cdot \text{s}} = 1.99$$

which is much, much lower than $10^3$, so the flow is laminar.

As for capillaries,

$$Re = \frac{\rho v R}{\eta} = \frac{(1.06 \times 10^3 \text{ kg/m}^3)(5 \times 10^{-4} \text{ m/s})(4 \times 10^{-6} \text{ m})}{4 \times 10^{-3} \text{ Pa} \cdot \text{s}} = 5 \times 10^{-4}$$

which also indicates laminar flow.
6. Measuring the viscosity of blood (2 pts)

The viscosity of blood depends on the red blood cell concentration: the higher the concentration, the higher the viscosity. A not-so-recent article described a simple apparatus for measuring the viscosity of blood (Journal of Non-Newtonian Fluid Mechanics, 94, 47-56, (2000)). The figure at right shows a schematic of the device. There are two cylindrical “riser tubes” that are each 3 mm in diameter. Connected between these two tubes is a capillary tube with an interior diameter of 0.80 mm and a length of 100 mm. The device is filled with blood through a stopcock such that the level of blood in one riser tube is higher than the level of blood in the other. This difference in heights creates a pressure difference across the capillary tube, which causes blood to flow through the capillary tube.

The levels of blood in the two riser tubes are measured as a function of time as shown in the graph below:

a) Given the initial height difference of 58 mm between the levels in the two riser tubes, calculate the initial pressure difference across the capillary tube.

The pressure difference is just the hydrostatic pressure difference between the top and bottom of a 58-mm column of blood:

\[ \Delta P = \rho gh \]

\[ = (1.06 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(0.058 \text{ m}) \]

\[ = 600 \text{ Pa} \]

This is equivalent to 4.5 torr.
b) From the graph, estimate the initial slope of the curves that show the height of blood as a function of time. What is the initial volumetric flow rate (in mL/s) for this sample of blood?

Using a straightedge, I estimated that if the initial slope of the curve for riser tube 1 were extended, it would reach zero at a time of around 50 seconds. So the slope is roughly

\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-86 \text{ mm}}{50 \text{ s}} = -1.7 \text{ mm/s}
\]

and the slope of the lower curve is +1.7 mm/s. This makes sense: the amount of fluid leaving riser tube 1 is exactly equal to the amount that enters riser tube 2, by continuity (the tubes have the same cross-sectional area). The slope tells us the speed of blood flow in the riser tubes. The initial volumetric flow rate in these tubes is then

\[
Q_i = AV_i = \pi \left(1.5 \times 10^{-3} \text{ m}\right)^2 \left(1.7 \times 10^{-3} \text{ m/s}\right) = 1.2 \times 10^{-8} \text{ m}^3/\text{s}.
\]

1 mL = 1 cm³ = 10⁻⁶ m³, so the flow rate is about \(0.012\) mL/s.

c) Assuming that the viscous resistance to flow can be neglected everywhere except in the capillary tube, estimate the viscosity of this sample of blood.

The flow through the capillary tube is laminar, so we can apply the Poiseuille equation to find the viscosity. (We better be able to, because Poiseuille is really the only thing that could possibly tell us anything about viscosity!) From the previous part, the initial flow rate is \(Q_i = 1.2 \times 10^{-8} \text{ m}^3/\text{s}\). The initial pressure difference across the tube is 600 Pa, from part a), with the higher pressure on the side of riser tube 1. So the viscosity of the blood is

\[
\eta = \frac{\pi R^4 \Delta P}{8QL} = \frac{\pi \left(4.0 \times 10^{-4} \text{ m}\right)^4 (600 \text{ Pa})}{8 \left(1.2 \times 10^{-8} \text{ m}^3/\text{s}\right) (0.10 \text{ m})} = 5.1 \times 10^{-3} \text{ Pa} \cdot \text{s}.
\]

This is close to the estimates for the viscosity of blood that we have seen in other problems, albeit a bit on the high side.

How did we know the flow was laminar? Intuitively, it’s a very thin tube and the flow rate seems pretty slow, but we could calculate the Reynolds number just as a sanity check. We can calculate the average speed of blood in the capillary tube based on the speed in the riser tubes and the continuity equation:

\[
\nu_{\text{cap}} = \frac{A_{\text{riser}}}{A_{\text{cap}}} \nu_{\text{riser}} = \left(\frac{r_{\text{riser}}}{r_{\text{cap}}}\right)^2 \nu_{\text{riser}} = \left(\frac{3.0}{0.80}\right)^2 (1.7 \text{ mm/s}) = 2.4 \text{ cm/s}
\]

Then the Reynolds number in the capillary tube is

\[
\text{Re} = \frac{\rho \nu R}{\eta} = \frac{(1.06 \times 10^3 \text{ kg} / \text{ m}^3)(0.024 \text{ m/s})(0.4 \text{ mm})}{5.1 \times 10^{-3} \text{ Pa} \cdot \text{s}} = 2
\]

Since 2 is much less than 1000, the flow is definitely laminar.