1. a) The standard form of an ODE initial value problem is \( \dot{x} = f(t, x), \ x(t_0) = x_0. \) Express the following ODE problem:
\[
\begin{align*}
\ddot{u} &= -v \frac{v}{1 + t^2} - \sin m \\
\ddot{v} &= +u \frac{u}{1 + t^2} + \cos m
\end{align*}
\]
in this form, given \( m \equiv \sqrt{\dot{u}^2 + \dot{v}^2}. \)

b) Given initial conditions \( u(0) = 1, \ v(0) = \dot{u}(0) = \dot{v}(0) = 0, \) solve this equation using the 4th order Runge-Kutta (RK4) scheme that you developed in class up to \( t = 1000 \) with timestep \( h = 1 \times 10^{-3}. \) Produce plots of \( u(t), \ v(t) \) and \( u(v). \) Describe the behavior of the system qualitatively.

c) Record the final value of \( u \) you obtained in the last example. Treating this as a reference value \( u_{ref}, \) find the largest value of \( h, \) \( h_{max} \) for which the error
\[
\epsilon(h) = \frac{|u_{ref} - u(t = 1000)|}{|u_{ref}|}
\]
remains less than 0.01%. Your estimate of \( h_{max} \) should be correct to within 10%.

2. A spacecraft containing three astronauts is re-entering Earth’s atmosphere. The spacecraft’s vertical acceleration as a function of time is
\[
\ddot{z} = -g + \frac{\alpha(t)}{m}
\]
Here \( z \) is altitude, \( g \) is Earth’s gravity (9.81 m/s/s), \( m \) is the spacecraft mass (10000 kg) and \( \alpha(t) \) is the friction force due to air. We can write \( \alpha(t) \) as
\[
\alpha = Ke^{-z/H} \dot{z}(t)^2
\]
where \( K = 2.0 \) and \( H = 10 \) km. The exponential term accounts for the fact that the atmosphere gets denser closer to the surface and hence air resistance greatly increases as the spacecraft descends.

a) Solve (4) analytically for the free-fall case (\( \ddot{z} = 0 \)) in order to write down the expression for the free-fall speed \( v_f = \dot{z} \) [you don’t need to worry about also calculating \( z(t) \) here]. What is the value of \( v_f \) at the surface?
b) Solve (4) numerically, again using the RK4 scheme, with initial condition \( z(0) = 100 \) km and \( v(0) = 0 \) m/s, over the interval \( 0 \leq t \leq 500 \) s. Plot the spacecraft altitude and velocity as a function of time. Using a numerical approach of your choice, determine the time at which the spacecraft hits the surface, \( t_{\text{impact}} \).

c) If you solved (4) correctly, you may have observed that the spacecraft’s speed on impact is not conducive to the continued health of the astronauts. Let’s now give the spacecraft a parachute. We can assume that after the parachute opens, \( K = 200 \). What is the new free-fall velocity near the surface in this case? By trial and error or a more sophisticated method, estimate the altitude at which the parachute should open in order to minimize the descent time while ensuring the impact speed is lower than 30 m/s. Do you think opening the parachute at this altitude will be a good idea in practice?

3. The Lotka-Volterra predator-prey model is a classic of mathematical ecology. In this model, there are rabbits with an infinite supply of food and foxes that prey on the rabbits. The system is modeled by the pair of nonlinear ordinary differential equations

\[
\begin{align*}
\dot{r} &= 2r - \alpha rf \\
\dot{f} &= -f + \alpha rf
\end{align*}
\]

where \( t \) is time, \( r(t) \) is the number of rabbits, \( f(t) \) is the number of foxes, and \( \alpha \) is a positive constant that captures how frequently foxes catch rabbits. Here, the \( \alpha rf \) term in the two equations differs by sign only, implying that every eaten rabbit leads to the birth of a new fox.

a) Using either RK4 or another method of your choice, compute the solution to (6) for \( r(0) = 300, \ f(0) = 15 \) and \( \alpha = 0.01 \) up to time \( t = 40 \). Make plots of \( r(t) \) and \( f(t) \) and a phase plane plot of \( r \) vs. \( f \) and describe the results. What is the maximum \( h \) value you can use while still obtaining reasonably accurate results? By inspection of the \( f(t) \) graph, estimate the period of the steady solution. Next, increase \( \alpha \) to 0.1. Plot the results and explain the difference in terms of the altered ecology of the system.

b) Next, calculate the period \( t_p \) numerically and plot it as a function of \( \alpha \) in the range \( 0.01 \leq \alpha \leq 0.1 \). To do this, you will need to come up with a way of determining when a cycle has completed. One good approach could be to focus on either \( r \) or \( f \) and think about the changing value of the derivative with time.

c) A more realistic model accounts for the fact that prey species can never have an infinite food supply. In this model, the equation for \( r \) is modified to

\[
\dot{r} = 2 \left( 1 - \frac{r}{R} \right) r - \alpha rf. \tag{7}
\]

Using \( R = 300, \alpha = 0.1 \) and an integration time \( t \) and step size \( h \) of your choice, solve this revised equation and compare the results to your previous calculation. What differences can you see? Predict the end state of your revised model analytically and compare the answer with your numerical results.