1. a) Write the analytical derivative of $f(x) = \cos^2(x)$ and calculate its value at $x = 1$.
   
b) Now approximate the derivative of $f(x) = \cos^2(x)$, using backward, forward and centered difference approximations. Calculate the value in each case as a function of the interval $h$ between $h = 0.0001$ and $h = 0.1$. Verify numerically that the convergence order is $O(h)$, $O(h)$ and $O(h^2)$ for the three cases, respectively.

2. Water flows into a cylindrical tank from an open top, and out of the tank at the bottom through a pipe. A differential equation for the water height $H$ in meters for the tank is

$$\frac{dH}{dt} = \left( 1 + \cos(\pi t/10) - 0.9\sqrt{H} \right) / 10$$

In this equation, the first term on the rhs represents inflow of water, the second term is cyclic, and the final term represents loss to the pipe.

a) Use the forward Euler method to write the above equation in finite-difference form. Use this expression to solve (1) numerically for a time interval of $\Delta t = 1$ s from $t = 0$ to $t = 200$ s, given $H(0) = 4$ m.

b) Write down the average steady-state value for $H$, $\langle H \rangle$, in the limit where $dH/dt = 0$. Plot the result alongside your numerical calculation from a).

c) Repeat the calculation in a) for $\Delta t = 10, 0.1, 0.01$ and 0.001 s. Plot all the results together alongside the previous $\Delta t = 1$ s case. Also plot $h(t = 200)$ as a function of $\Delta t$, and comment on the result.
3. One integral with particularly interesting behavior is

\[ I = \int_a^1 f(t)dt \]  
where

\[ f(t) = t^{-1} \cos(t^{-1} \log t). \]

a) Use the composite trapezoidal rule to integrate (2) numerically given \( a = 0.15 \). Test the convergence of your answer as the number of points is increased from 10 to \( 10^4 \). Do the same thing with the composite midpoint rule.

b) Now use \texttt{linspace} to create a fine grid for \( t \) (at least \( 10^4 \) points) and do

\texttt{plot(x,f(t),'k'); axis([0 1 -50 50])}

with \( f(t) \) defined as in (3) to plot \( f(t) \). Using trapezoidal integration with \( 10^3 \) points, plot \( I(a) \), with \( a \) on a log scale from \( 10^{-8} \) to 0.1. What goes wrong with your calculation as \( a \to 0 \)? Do you think your answer will converge if you keep increasing the number of points?

c) One successful strategy for integrating \( f(t) \) in the domain \([0,1]\) involves first changing variables to \( u = t^{-1} \), so

\[ I = \int_1^\infty \frac{\cos(u \log u)}{u} du, \]

and then to \( x = u \log u \), such that

\[ I = \int_0^\infty \frac{\cos(x)}{x + u(x)} dx. \]

When this integral is performed correctly, the result is approximately \( I_c \approx 0.32336743 \). Can you numerically integrate (5) to show convergence to this result? Hint: you will need to make use of Newton’s method or a similar root-finding algorithm. To demonstrate graphically that your result is converging, you can plot the cumulative integral

\[ I(b) = \int_0^b \frac{\cos x}{x + u(x)} dx. \]

from \( b = 0 \) to \( b = 200 \) or so and compare the result to the value of \( I_c \).