Last time we covered...

Math 122

- G a group, $H \triangleleft G$ subgroup
  - Note: $G \rightarrow G/H$ is always defined, but $H$ normal $g \mapsto gH$ makes $G/H$ into a group.

- Thus, Assume $H$ is normal in $G$ then the function $G/H \times G/H \rightarrow G/H$ $(gH, g' H) \mapsto gHg'^{-1}H$ is well-defined and makes $G/H$ a group.

- Def. A subgroup $H$ is called normal in $G$ when $gHg^{-1} = H$ for all $g \in G$.
  - ie $\{ x \in G | x = ghg^{-1}, h \in H \}$ (so $g \in G, h \in G, ghg^{-1} \in H$)

Defn. When $H$ is normal we write $H \triangleleft G$, $G/H$ (when $H \triangleleft G$) is called the quotient gp of $G$ by $H$.

If that this multiplication fn is well defined

Want to prove: if $g_1H = g_1'H$ \iff $\exists h_1 \text{ s.t. } g_1 = g_1'h_1$,

$g_2H = g_2'H$ \iff $\exists h_2 \text{ s.t. } g_2 = g_2'h_2$,

then $g_1g_2H = g_1'g_2'H$ \textbf{Show:} $g_1g_2 = g_1'g_2'h$ for some $h \in H$
We know \( g_1 g_2 = (g_1', h_1)(g_2', h_2) \)
\[ = g_1'(g_2' h_1 (g_2')^{-1}) g_2' h_2 \]
\[ = g_1' g_2' h_1 h_2 \]
\[ = h \quad \text{for } h \in H \]

Since \( H \triangleleft G \): e.g. \( g H g^{-1} = H \)
\[ \forall h_1, h \in H, \text{s.t.} \]
\[ h_1 = g_2' h_1 (g_2')^{-1} \]
\[ \{ H c G_2 H (g_2')^{-1} \} \]

**Defn.** Fix gp. hom. \( \phi : G \to L \)
The kernel of \( \phi \) is the set
\[ \ker \phi = \{ g \in G | \phi(g) = e_L \} \]

**Excs.** Fix \( \phi : G \to L \)
Prove: \( \phi \) injection
\[ \Downarrow \]
\[ \ker \phi = \{ e_G \} \]

**Pf.** (\( \forall \)) \( \phi \) injective \( \Rightarrow \)
\[ \phi(g_1) = \phi(g_2) \]
\[ \Rightarrow g_1 = g_2 \]
\[ \Rightarrow \text{we know } \phi(e_G) = e_L \]
so \( \phi(g_1) = e_L \Rightarrow g_1 = e_G \)

\[(\exists) \quad \text{Suppose } \phi(g_1) = \phi(g_2) \]
\[ \Rightarrow \phi(g_2)^{-1} \phi(g_1) = e_L \]
\[ \Rightarrow \phi(g_2^{-1} g_1) = e_L \]
\[ \Rightarrow g_2^{-1} g_1 \in \ker \phi \]
\[ \Rightarrow g_2^{-1} g_1 = e_G \]
\[ \Rightarrow g_2 = g_1 \]

**Defn.** Fix \( \phi : G \to L \) gp hom.
Then the **image** of \( \phi \) is
\[ \text{Im}(\phi) = \{ l \in L | l = \phi(g) \text{ for some } g \in G \} \]
Ex 5. (1) \( \text{Im } \phi \subset L \) is a subgroup
(2) \( \ker \phi \subset L \)
(3) \( \ker \phi \triangleleft G \)

Proof. (1) Need to show \( \text{Im } \phi \cong e_L \) is closed under inv. and mult.
\[
\begin{align*}
L &= \phi(g) \implies L^{-1} = \phi(g^{-1}) = \phi(g)^{-1} \\
L_L &= \phi(g_1) \implies L_L L_2 = \phi(g_1) \phi(g_2) \\
&= \phi(g_1 g_2)
\end{align*}
\]
(2) similar
(3) Fix \( g \in G \), \( k \in \ker \phi \)
\[
\begin{align*}
\phi(gk^{-1}) &= \phi(gk) \phi(k^{-1}) \\
&= \phi(g) e_L \phi(g)^{-1} \\
&= e_L
\end{align*}
\]

Remark. Every subgroup \( L' \subset L \) is the image of some group homomorphism.
\[
L' \rightarrow L \\
\begin{cases}
L & \longrightarrow L \\
l & \mapsto \phi(l)
\end{cases}
\]
the inclusion

Thus. (1st isomorphism thm.)
- Fix a homom \( \phi: G \rightarrow L \)
- There exists a natural isomorphism \( G/\ker \phi \cong \text{Im } \phi \) and this is an isomorphism.

Thus Fix \( \phi: G \rightarrow L \) a gp. hom. Assume \( KCG \) is a subgroup s.t. \( KC \triangleleft \ker \phi \)

Thus \( G \twoheadrightarrow \phi \rightarrow L \)

Moreover if \( KCG \) then \( \phi \) is a group homomorphism.

Ex. \( G = \mathbb{Z} \)
\[
\begin{array}{c}
G = \mathbb{Z} \\
\phi: \mathbb{Z} \rightarrow L \\
\downarrow \phi \\
\mathbb{Z}/\ker \phi \\
\downarrow \mathbb{Z}/n \mathbb{Z}
\end{array}
\]
so the fn \( \phi \) actually didn't care about the integer, it just cared about the integer mod \( n \)
pf of \(\exists! \psi\) then

Note that if \(\psi\) exists, it has to be unique because it is a surjection.

Define \(\psi\) to be: \(g_1 k \mapsto \phi(g)\).

Need to show: If \(g_1 k = g_2 k\), \(\psi(g_1) = \psi(g_2)\).

\(g_1 k = g_2 k \Rightarrow g_1 = g_2 k\) for \(k \in K\).

\[ \Rightarrow \phi(g_1) = \phi(g_2 k) \]
\[ = \phi(g_2) \phi(k) \]
\[ = \phi(g_2) e_1 \]
\[ = \phi(g_2) \]

\(\square\)

When \(K \vartriangleleft G\) need to show \(\forall g \in G, \psi(g_1 k g_2 k) = \psi(g_1 k) \psi(g_2)\).

\(\psi(g_1 k g_2 k) = (\psi(g_1 g_2 k) = \phi(g_1 g_2)\)

\[ = \phi(g_1) \phi(g_2) = (\psi(g_1 k) \psi(g_2 k)\)

"Universal property of quotient groups."