Fix a group action of $G$ on $X$ ($G \times X \rightarrow X$)
Def: A subset $I \subset X$ is called an orbit of this action if the
following hold
1. $I \neq \emptyset$
2. $\forall y, y_2 \in I, \exists g \in G$ s.t. $gy_1 = y_2$ (you can get between any
two elements by the group action)
3. $\forall y \in I, \exists g \in G, gy \in I$

Prop. Let $I$ be an orbit. Then $\forall x \in I, I = \Theta_x$ ($\Theta_x = \{ y \in X | y = gx \}$)
Proof. $I \subseteq \Theta_x$ by (2): If $y \in I, y = gx$
$\Theta_x \subseteq I$ by (3): Set $x = y$ then $gy \in I \forall g \in G$

Corollary. (i) Every element $x \in X$ is contained in some orbit ($\Theta_x$)
(ii) If $I, J$ are orbits, either they have no intersection $I \cap J = \emptyset$ or they are equal $I = J$
(iii) $\Theta_x = \Theta_y \iff \exists g \in G$ s.t. $gx = y$

(Trick) Question: For what $H$ does the fn
\[ G/\Delta \times G/H \rightarrow G/H \]
$g_1 H, g_2 H \mapsto (g_1 g_2) H$ make $G/\Delta$ a group?

My thoughts:
\[ g_1 H = (g_1, H) \]
\[ g_1^{-1} h H = g_1^{-1} h g_1 H \]
\[ H = \{(g_1 g_2 g_1^{-1} g_2^{-1}) H \}
\]

\[ g_1^{-1} h g_1 H = g_1^{-1} H g_1 H \]
\[ h g_1 H = g_1 H g_1 H \]
\[ h g_1 H = g_1 H g_1 H \]

Problem: Suppose $g_1 H = g_1 H$ and $g_2 H = g_2 H$. Does $g_1 g_2 H = g_1 g_2 H$?
My thoughts:
\[ g_1 g_2 h = g_1 g_2 \]
\[ h = g_2 g_1^{-1} g_1 g_2 \]
\[ g_1 = h g_1, g_2 = g_2 h \]
\[ h = g_2 g_1^{-1} h g_1 h g_2 \]
\[ g_1 h g_2 = h g_1 h g_2 \]
Prop. Suppose $H$ satisfies $\forall g \in G$, $gHg^{-1} = H$. Then the problem has an affirmative answer.

Ex. • All abelian groups $G$, you can make any subgroup $H$ work
• For any group $G$, $H=\{e\}$ works and $G/H$ isomorphic to $G$.

**Proof of proposition**

The idea: Assume $g_2=g_2$.

Note: $g_1H=g_2 \Rightarrow g_1=g_2h$, for some $h \in H$.

To show $g_1g_2H=g_2g_2H$, want to show that for $h \in H$,

$$g_1g_2h \in g_2g_2H$$

Since $g_1 \in g_1H$,

$$g_1g_2h = g_1h \cdot g_2h$$

$$= g_1(g_2h \cdot g_2h^{-1})g_2h$$

$$= g_1g_2h \cdot g_2g_2H$$

**Def.** A subgroup $H \subseteq G$, is called "normal" (in $G$) iff $\forall g \in G$, $gHg^{-1} = H$.

So we answered the question!!

Thus, if $H$ is a normal subgroup of $G$, then the fn

$$G/H \times G/H \rightarrow G/H$$

$$(g_1H, g_2H) \mapsto g_1g_2H$$

makes $G/H$ into a group.

Note: $\exists$ fn $G \rightarrow G/H$. This is a group homomorphism because $g \mapsto gH$.

**Pf.** Call the fn $q: G \rightarrow G/H$.

$$q(g_1g_2) = g_1g_2H = g_1Hg_2H = q(g_1)q(g_2)$$

We are "collapsing $H$":

**Thm.** $q^{-1}(e_{G/H})$ = \{ $g \in G$ s.t. $q(g) = e_{G/H}$ \}

= \{ $g \in G$ s.t. $q(g) \in H$ \}

= $H$. 


In general, when we have homomorphisms between groups, we wanna know what gets collapsed.

**Def.** Let $\phi: G \to K$ be a group homomorphism. The kernel of $\phi$, denoted $\ker(\phi)$, is the set $\ker(\phi) = \{g \in G | \phi(g) = e_K\}$.

**Thm.** A subgroup $H \triangleleft G$ is normal (in $G$) iff $H = \ker(\phi)$ for some group homomorphism $\phi: G \to K$. 