Lecture Four: Equivalence relations!

Hiro can't attend lecture four, but it's perfect timing for a topic called *equivalence relations*.

This is an idea that you should get your hands dirty with, so this lecture will be completely exercise-based.

Once you get this print-out, you can start talking with your classmates (or work on your own, whatever you prefer) and get cracking!

Notation

Given a set S, the notation $s \in S$ means that s is an element of S. $s \notin S$ means s is *not* an element in S.

The notation $A \subset B$ means that A is a subset of B.

Given a pair of sets S and T (where the two sets may be equal) the *direct product* of S and T is denoted $S \times T$. An element of $S \times T$ is an ordered pair (s, t), where $s \in S$ and $t \in T$.

The symbol $\mathbb{Z}$ stands for the set of all integers.

Finally, the symbol ":=" means we define something to equal another thing. For example,

$$2\mathbb{Z} := \{n \in \mathbb{Z} \text{ such that } n \text{ is even } \}$$

means we define the symbol $2\mathbb{Z}$ to stand for the collection of all even numbers.

1. Equivalence relations

We will encounter equivalence relations when we consider *quotients* of groups. (Whatever that means!)

An equivalence relation is a (very) formal way of realizing the question: "Hey, when should we consider two things to be the same?"

Definition 1.1. Let S be a set. A *relation* on S is a choice of subset

$$R \subset S \times S.$$

A relation R is called an *equivalence relation* if the following three properties are satisfied:

(1) (Reflexivity) R contains the diagonal of S. That is, for every $x \in S$, the element (x, x) is contained in R.

(2) (Symmetry) If (x, y) is in R, then (y, x) is in R.

(3) (Transitivity) If (x, y) and (y, z) are both in R, then (x, z) is in R.

If R is an equivalence relation and $(x, y) \in R$, we say that x *is related to* y. (Note that by symmetry, if x is related to y, then y is related to x.)

REMARK 1.2. If $(x, y) \in R$, later in the course, you should think of this as code for "pretend that x and y are the same." The reason we take R to be a subset of $S \times S$ is simply because picking out an element of $S \times S$ is the same thing as picking out a(n ordered) pair of elements in S.

EXAMPLE 1.3. Let $S = \mathbb{R}$. If $R \subset \mathbb{R} \times \mathbb{R}$ is the graph of some function $f : \mathbb{R} \to \mathbb{R}$, then R is an equivalence relation if and only if f is the identity function.

(a) For any set S, let R be the diagonal. That is, $R \subset S \times S$ consists exactly of elements of the form (x, x), for every $x \in S$. Show R is an equivalence relation.

(b) Let $S = \mathbb{Z}$ be the set of integers. Fix a non-zero integer n. Declare R to be the set of all pairs (x, y) such that $x - y$ is divisible by n. Show that R is an equivalence relation. (Recall that an integer z is said to be *divisible by* n if $z = an$ for some integer a. In particular, both z and a could be negative.)

(c) Let $S = \{0, 1, \ldots, 24\}$ be the set of integers from 0 to 24. Let $R \subset S \times S$ be the set of all pairs (x, y) such that $x - y$ is divisible by 12. Show that R is an equivalence relation. Do you see this at all in your daily life? Maybe hanging on the wall?

2. Fun with equivalence classes

Let $R \subset S \times S$ be an equivalence relation. We will write

$$x \sim y$$

if and only if (x, y) is in R. When we want to make the dependence on R explicit, we may sometimes decorate our $\sim$ with the symbol R as follows:

$$x \sim_R y.$$

Definition 2.1. Fix a set S and an equivalence relation $R \subset S \times S$. For any $x \in S$, we define a set $[x]$ as follows:

$$[x] := \{y \in S \text{ such that } x \sim y\}.$$

We say that $[x]$ is the *equivalence class of* x.

Note that $[x]$ is a subset of S. Note also that x is an element of $[x]$. It may help to vocalize $[x]$ as "bracket x" when you talk to your friends.

(a) Let x and y be two elements of S. Show that either (i) the sets $[x]$ and $[y]$ are equal, or (ii) the sets $[x]$ and $[y]$ have no intersection. (This means that any equivalence relation R "breaks up" S into a disjoint union of sets.)

Definition 2.2. If R is an equivalence relation on a set S, we let

$$S/\sim$$

denote the collection of equivalence classes determined by R.

Remark 2.3. Confusingly, $S/\sim$ is a set of sets! That is, an element of $S/\sim$ is a set. Later in the class, it will be *very* convenient, and less cumbersome, to think of $S/\sim$ simply as a set (ignoring the truth that its elements themselves form sets). Paradoxically, you should think of $S/\sim$ as obtained by "collapsing" any two related elements into a single gadget, called their equivalence class.

(b) Let $S = \mathbb{Z}$, and let R be the equivalence relation from problem 1(b), with $n = 4$. It turns out there are exactly 4 disjoint equivalence classes determined by R—write them all out. That is, $\mathbb{Z}/\sim$ is a collection of four sets. Write out all four sets.

(c) More generally, for any non-zero n, and for the relation R from problem 1(b), show that $\mathbb{Z}/\sim$ is in bijection with the set of integers $\{0, \ldots, n-1\}$ between 0 and $n-1$, inclusive. (In particular, $\mathbb{Z}/\sim$ is a collection of n sets.)

3. Orbits are equivalence classes

Let X be a set, and let $G \times X \to X$ be a group action.

(a) Let $R \subset X \times X$ consist of those pairs (x, y) such that $y = gx$ for some $g \in G$. Show that R is an equivalence relation.

(b) Show that the equivalence class $[x]$ is equal to the orbit $\mathcal{O}_x$.

(c) Infer that $X/\sim$ is the same thing as the set of orbits of the group actions.

4. Conjugation as an example to practice if you have time

(a) Fix an integer $n \geq 1$. Let $S = M_n(\mathbb{R})$ be the set of n-by-n matrices with real entries. Let $R \subset S \times S$ consists of pairs (A, B) such that there exists some invertible n-by-n matrix C for which $A = CBC^{-1}$. Show that R is an equivalence relation.

(b) More generally, fix a group G and set $X = G$. We define a function

$$\mu : G \times G \to G$$

by defining $\mu(g, x) := gxg^{-1}$ for any $g, x \in G$. Show that μ is a left action of G on itself. This is called the *conjugation action* of G on itself.

(c) Show that $\mu'(x, g) := g^{-1}xg$ defines a right action of G on itself.

(d) Show that if G is abelian, each orbit of the conjugation action has only one element.

5. Some challenges in case you want them!

(a) Let $S = \mathbb{R}$ and $R \subset S \times S$ be the collection of pairs (t_1, t_2) such that $t_2 - t_1$ is a multiple of 2π. Is there a natural shape you want to associate to the set $\mathbb{R}/\sim$?

(b) Let $S = \mathbb{R}^2$ and $R \subset S \times S$ be the collection of pairs $((x_1, y_1), (x_2, y_2))$ such that $x_1 - x_2$ and $y_1 - y_2$ are both integers. Is there a natural shape you want to associate to $S/\sim$? What does Pacman have to do with it? (Caution: This shape is *not* Pacman.)

(c) Here's a much harder challenge. Let $S = \mathbb{C}^2 \setminus \{0\}$ be the collection of pairs of complex numbers (z_1, z_2), with $(0, 0)$ thrown out. Let's say that two elements (z_1, z_2) and (w_1, w_2) of S are related if there exists a (non-zero) complex number a such that $(az_1, az_2) = (w_1, w_2)$. Is there a natural shape you want to associate to $S/\sim$? (It is a shape you have definitely seen before.)