MATHEMATICS E-23a, Fall 2016
Linear Algebra and Real Analysis I
Module #3, Week 3
Differentiability, Newton's method, inverse functions

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Reading

- Hubbard, section 1.7 (you have already read most of this)
- Hubbard, sections 1.8 and 1.9 (computing derivatives and differentaibility)
- Hubbard, section 2.8 page 233-235 and page 246. (Newton's method)
- Hubbard, section 2.10 up through page 264. (inverse function theorem)

Recorded Lectures

- November 17, 2015 (watch on November 15)
- November 19, 2015 (watch on November 17)

Proofs to present in section or to a classmate who has done them.

- 11.1 Let $U \subset \mathbb{R}^n$ be an open set, and let f and g be functions from U to $\mathbb{R}$. Prove that if f and g are differentiable at $\mathbf{a}$ then so is fg, and that

$$[\mathbf{D}(fg)(\mathbf{a})] = f(\mathbf{a})[\mathbf{D}g(\mathbf{a})] + g(\mathbf{a})[\mathbf{D}f(\mathbf{a})].$$

- 11.2 Using the mean value theorem, prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ that are continuous at $\mathbf{a}$, it is differentiable at $\mathbf{a}$ and its derivative is the Jacobian matrix $\begin{bmatrix} D_1 f(\mathbf{a}) & D_2 f(\mathbf{a}) \end{bmatrix}$.

R Scripts

- Script 3.3A-ComputingDerivatives.R
  Topic 1 - Testing for differentiability
  Topic 2 - Illustrating the derivative rules
- Script 3.3B-NewtonsMethod.R
  Topic 1 - Single variable
  Topic 2 - 2 equations, 2 unknowns
  Topic 3 - Three equations in three unknowns
- Script 3.3C-InverseFunction.R
  Topic 1 - A parametrization function and its inverse
  Topic 2 - Visualizing coordinates by means of a contour plot
  Topic 3 - An example that is economic, not geometric

1 Executive Summary

1.1 Definition of the derivative

- Converting the derivative to a matrix
  The linear function $f(h) = mh$ is represented by the 1×1 matrix $[m]$.
  When we say that $f'(a) = m$, what we mean is that the function $f(a+h) - f(a)$ is well approximated, for small h, by the linear function mh. The error made by using the approximation is a "remainder" $r(h) = f(a+h) - f(a) - mh$. If f is differentiable, this remainder approaches 0 faster than h, i.e.

  $$\lim_{h \to 0} \frac{r(h)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a) - mh}{h} = 0.$$

  This definition leads to the standard rule for calculating the number m,

  $$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Extending this definition to $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  A linear function $L(\vec{\mathbf{h}})$ is represented by an $m \times n$ matrix.
  When we say that $\mathbf{f}$ is differentiable at $\mathbf{a}$, we mean that the function $\mathbf{f}(\mathbf{a} + \vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a})$ is well approximated, for any $\vec{\mathbf{h}}$ whose length is small, by a linear function L, called the derivative $[\mathbf{Df}(\mathbf{a})]$.
  The error made by using the approximation is a "remainder"
  $\mathbf{r}(\vec{\mathbf{h}}) = \mathbf{f}(\mathbf{a} + \vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})](\vec{\mathbf{h}})$.
  $\mathbf{f}$ is called differentiable if this remainder approaches 0 faster than $|\vec{\mathbf{h}}|$, i.e.

  $$\lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|} \mathbf{r}(\vec{\mathbf{h}}) = \lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|} (\mathbf{f}(\mathbf{a} + \vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})](\vec{\mathbf{h}})) = \mathbf{0}.$$

  In that case, $[\mathbf{Df}(\mathbf{a})]$ is represented by the Jacobian matrix $[\mathbf{Jf}(\mathbf{a})]$.

  Proof: Since L exists and is linear, it is sufficient to consider its action on each standard basis vector. We choose $\vec{\mathbf{h}} = t\vec{\mathbf{e_i}}$ so that $|\vec{\mathbf{h}}| = t$. Knowing that the limit exists, we can use any sequence that converges to the origin to evaluate it, and so

  $$\lim_{t \to 0} \frac{1}{t} (\mathbf{f}(\mathbf{a} + t\vec{\mathbf{e_i}}) - \mathbf{f}(\mathbf{a}) - tL\vec{\mathbf{e_i}})) = 0? \text{ and } L(\vec{\mathbf{e_i}}) = \lim_{t \to 0} \frac{1}{t} (\mathbf{f}(\mathbf{a} + t\vec{\mathbf{e_i}}) - \mathbf{f}(\mathbf{a}))$$

  What is hard is proving that f is differentiable – that L exists – since that requires evaluating a limit where $\vec{\mathbf{h}} \to \vec{\mathbf{0}}$. Eventually we will prove that f is differentiable at $\mathbf{a}$ if all its partial derivatives are continuous there.

1.2 Proving differentiability and calculating derivatives

In every case $\mathbf{f}$ is a function from U to $\mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$.

- $\mathbf{f}$ is constant: $\mathbf{f} = \mathbf{c}$. Then $[\mathbf{Df}(\mathbf{a})]$ is the zero linear transformation, since

$$\lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{f}(\mathbf{a}+\vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})]\vec{\mathbf{h}}) = \lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{c} - \mathbf{c} - \vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

- $\mathbf{f}$ is affine: a constant plus a linear function, $\mathbf{f} = \mathbf{c} + L$. $[\mathbf{Df}(\mathbf{a})] = L$, since

$$\lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{f}(\mathbf{a}+\vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})]\vec{\mathbf{h}}) = \lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{c}+L(\mathbf{a}+\vec{\mathbf{h}}) - (\mathbf{c}+L(\mathbf{a})) - L(\vec{\mathbf{h}})) = 0.$$

$\mathbf{f}$ has differentiable components: if $\mathbf{f} = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$: then $\mathbf{Df}(\mathbf{a}) = \begin{bmatrix} \mathbf{D}f_1(\mathbf{a}) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{D}f_n(\mathbf{a}) \end{bmatrix}$

.

- $\mathbf{f} + \mathbf{g}$ is the sum of two functions $\mathbf{f}$ and $\mathbf{g}$, both differentiable at $\mathbf{a}$.
  The derivative of $\mathbf{f} + \mathbf{g}$ is the sum of the derivatives of $\mathbf{f}$ and $\mathbf{g}$. (easy to prove)

- $f\mathbf{g}$ is the product of scalar-valued function f and vector-valued $\mathbf{g}$, both differentiable. Then
  $[\mathbf{D}(f\mathbf{g})(\mathbf{a})]\vec{\mathbf{v}} = f(\mathbf{a})([\mathbf{Dg}(\mathbf{a})]\vec{\mathbf{v}}) + ([\mathbf{D}f(\mathbf{a})]\vec{\mathbf{v}})\mathbf{g}(\mathbf{a})$.

- $\mathbf{g}/f$ is the quotient of vector-valued function $\mathbf{g}$ and scalar-valued f, both differentiable, and $f(\mathbf{a}) \neq 0$. Then

$$[\mathbf{D}(\frac{\mathbf{g}}{f})(\mathbf{a})]\vec{\mathbf{v}} = \frac{[\mathbf{Dg}(\mathbf{a})]\vec{\mathbf{v}}}{f(\mathbf{a})} - \frac{([\mathbf{D}f(\mathbf{a})]\vec{\mathbf{v}})\mathbf{g}(\mathbf{a})}{(f(\mathbf{a})^2}.$$

- $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets, and $\mathbf{a}$ is a point in U at which we want to evaluate a derivative.

  $\mathbf{g} : U \to V$ is differentiable at $\mathbf{a}$, and $[\mathbf{Dg}(\mathbf{a})]$ is an $m \times n$ Jacobian matrix.

  $\mathbf{f} : V \to \mathbb{R}^p$ is differentiable at $\mathbf{g}(\mathbf{a})$, and $[\mathbf{Df}(\mathbf{g}(\mathbf{a}))]$ is a $p \times m$ Jacobian matrix.

  The chain rule states that $[\mathbf{D}(\mathbf{f} \circ \mathbf{g})(\mathbf{a}))] = [\mathbf{Df}(\mathbf{g}(\mathbf{a}))] \circ [\mathbf{Dg}(\mathbf{a})]$.

- The combined effect of all these rules is effectively that if a function is defined by well-behaved formulas (no division by zero), it if differentiable, and its derivative is represented by its Jacobian matrix.

1.3 Connection between Jacobian matrix and derivative

- If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is defined on an open set $U \in \mathbb{R}^n$, and

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{x}) \\ ... \\ f_m(\mathbf{x}) \end{pmatrix}$$

  the Jacobian matrix $[\mathbf{Jf}(\mathbf{x})]$ is made up of all the partial derivatives of $\mathbf{f}$:

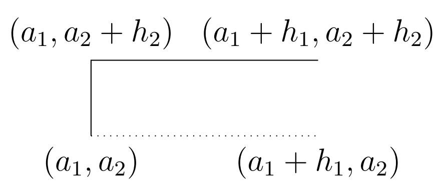
$$[\mathbf{Jf}(\mathbf{a})] = \begin{bmatrix} D_1 f_1(\mathbf{a})....D_n f_1(\mathbf{a}) \\ ... \\ D_1 f_m(\mathbf{a})....D_n f_m(\mathbf{a}) \end{bmatrix}$$

- We can invent pathological cases where the Jacobian matrix of f exists (because all the partial derivatives exist), but the function f is not differentiable. In such a case, using the formula

$$\nabla_{\vec{\mathbf{v}}} f(\mathbf{a}) = [\mathbf{J}f(\mathbf{a})]\vec{\mathbf{v}}$$

  generally gives the wrong answer for the directional derivative! You are trying to use a linear approximation where none exists.

- Using the Jacobian matrix of partial derivatives to get a good affine approximation for $f(\mathbf{a}+\vec{\mathbf{h}})$ is tantamount to assuming that you can reach the point $\mathbf{a}+\vec{\mathbf{h}}$ by moving along lines that are parallel to the coordinate axes and that the change in the function value along the solid horizontal line is well approximated by the change along the dotted horizontal line. With the aid of the mean value theorem, you can show that this is the case if (proof 11.2) the partial derivatives of f at $\mathbf{a}$ are continuous.



1.4 Newton's method – one variable

Newton's method is based on the tangent-line approximation. Function f is differentiable. We are trying to solve the equation $f(x) = 0$, and we have found a value a_0 that is close to the desired x. So we use the best affine approximation $f(x) \approx f(a_0) + f'(x_0)(x - a_0)$.
Then we find a value a_1 for which this tangent-line approximation equals zero. $f(a_0) + f'(x_0)(a_1 - a_0) = 0$, and $a_1 = a_0 - f(a_0)/f'(a_0)$.
When $f(a_0)$ is small, $f'(a_0)$ is large, and $f'(a_0)$ does not change too rapidly, a_1 is a much improved approximation to the desired solution x. Details, for which Kantorovich won the Nobel prize in economics, are in Hubbard.

1.5 Newton's method – more than one variable

Example: we are trying to solve a system of n nonlinear equations in n unknowns, e.g.
$x^2 e^y - \sin(y) - 0.3 = 0$
$\tan x + x^2 y^2 - 1 = 0.$
Ordinary algebra is no help – there is no nonlinear counterpart to row reduction.
U is an open subset of $\mathbb{R}^n$, and we have a differentiable function $\vec{\mathbf{f}}(\mathbf{x}) : U \to \mathbb{R}^n$.
In the example, $\vec{\mathbf{f}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 e^y - \sin(y) - 0.3 \\ \tan x + x^2 y^2 - 1 \end{pmatrix}$, which is differentiable.
We are trying to solve the equation $\vec{\mathbf{f}}(\mathbf{x}) = \vec{\mathbf{0}}$.
Suppose we have found a value $\mathbf{a_0}$ that is close to the desired $\mathbf{x}$.
Again we use the best affine approximation

$$\vec{\mathbf{f}}(\mathbf{x}) \approx \vec{\mathbf{f}}(\mathbf{a_0}) + [\mathbf{D}\tilde{\mathbf{f}}(\mathbf{a_0})](\mathbf{x} - \mathbf{a_0}).$$

We set out to find a value $\mathbf{a_1}$ for which this affine approximation equals zero.

$$\vec{\mathbf{f}}(\mathbf{a_0}) + [\mathbf{D}\tilde{\mathbf{f}}(\mathbf{a_0})](\mathbf{a_1} - \mathbf{a_0}) = \vec{\mathbf{0}}$$

This is a linear equation, which we know how to solve!
If $[\mathbf{D}\tilde{\mathbf{f}}(\mathbf{a_0})]$ is invertible (and if it is not, we look for a better $\mathbf{a_0}$), then

$$\mathbf{a}_1 = \mathbf{a}_0 - [\mathbf{D}\tilde{\mathbf{f}}(\mathbf{a_0})]^{-1} \vec{\mathbf{f}}(\mathbf{a}_0).$$

Iterating this procedure is the best known for solving systems of nonlinear equations. Hubbard has a detailed discussion (which you are free to ignore) of how to use Kantorovich's theorem to assess convergence.

1.6 The inverse function theorem – short version

For function $f : [a, b] \to [c, d]$, we know that if f is strictly increasing or strictly decreasing on interval $[a, b]$, there is an inverse function g for which $g \circ f$ and $f \circ g$ are both the identity function. We can find $g(y)$ for a specific y by solving $f(x) - y = 0$, perhaps by Newton's method. If $f(x_0) = y_0$ and $f'(x_0) \neq 0$, we can prove that g is differentiable at y_0 and that $g'(y_0) = 1/f'(x_0)$.
"Strictly monotone" does not generalize, but "nonzero $f'(x_0)$" generalizes to "invertible $[\mathbf{Df}(\mathbf{x}_0)]$." Start with a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ whose partial derivatives are all continuous, so that we know that it is differentiable everywhere. Choose a point $\mathbf{x}_0$ where the derivative $[\mathbf{Df}(\mathbf{x}_0)]$ is an invertible matrix. Set $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Then there is a differentiable local inverse function $\mathbf{g} = \mathbf{f}^{-1}$ such that

- $\mathbf{g}(\mathbf{y}_0) = \mathbf{x}_0$.
- $f(\mathbf{g}(\mathbf{y})) = \mathbf{y}$ if $\mathbf{y}$ is close enough to $\mathbf{y}_0$.
- $[\mathbf{Dg}(\mathbf{y})] = [\mathbf{Df}(\mathbf{g}(\mathbf{y}))]^{-1}$ (follows from the chain rule)

2 Lecture outline

1. The derivative as a linear transformation

   When we say that function f is differentiable at $x = a$ and that $f'(a) = m$, what we mean is that the function $f(a+h) - f(a)$ is well approximated, for small h, by a linear function $L(h) = mh$, where $m = f'(a)$.

   Show how this idea can be viewed as a "tangent-line approximation" to $f(x)$ for x near to a.

   In the single-variable case, we usually think of the derivative $f'(a)$ as just a number, not a linear function of an increment h, but that view will not generalize to derivatives in $\mathbb{R}^n$. Here is a view of single-variable calculus that generalizes correctly.

   Any linear function $L(h) = mh$ is represented by the 1×1 matrix $[m]$, which in turn is represented by the real number m.

   The error made by using the tangent-line approximation $f(a+h) - f(a) = f'(a)h$ is a "remainder"

   $$r(h) = f(a+h) - f(a) - f'(a)h.$$

   If f is differentiable, this remainder approaches 0 faster than h, i.e.

   $$\lim_{h \to 0} \frac{r(h)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

   This definition leads to the standard rule for calculating the number $f'(a)$,

   $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

   What mathematical object represents a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$?

2. Extending this definition to $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
A linear function $L(\vec{\mathbf{h}})$ is represented by an $m \times n$ matrix.

What matrix represents the linear function

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ 3x_1 - x_2 \end{pmatrix}?$$

When we say that $\mathbf{f}$ is *differentiable* at $\mathbf{a}$, we mean that the function $\mathbf{f}(\mathbf{a} + \vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a})$ is well approximated, for any $\vec{\mathbf{h}}$ whose length is small, by a linear function L, called the derivative $[\mathbf{Df}(\mathbf{a})]$.

The error made by using the approximation is a "remainder"
$\mathbf{r}(\vec{\mathbf{h}}) = \mathbf{f}(\mathbf{a} + \vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})](\vec{\mathbf{h}})$.
$\mathbf{f}$ is called differentiable if this remainder approaches 0 faster than $|\vec{\mathbf{h}}|$, i.e.

$$\lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|} \mathbf{r}(\vec{\mathbf{h}}) = \lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|} (\mathbf{f}(\mathbf{a} + \vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})](\vec{\mathbf{h}})) = \mathbf{0}.$$

In that case, $[\mathbf{Df}(\mathbf{a})]$ is represented by the Jacobian matrix $[\mathbf{Jf}(\mathbf{a})]$.

Proof: Since L exists and is linear, it is sufficient to consider its action on each standard basis vector. We choose $\vec{\mathbf{h}} = t\vec{\mathbf{e_i}}$ so that $|\vec{\mathbf{h}}| = t$. Knowing that the limit exists, we can use any sequence that converges to the origin to evaluate it, and so

$$\lim_{t \to 0} \frac{1}{t} (\mathbf{f}(\mathbf{a} + t\vec{\mathbf{e_i}}) - \mathbf{f}(\mathbf{a}) - tL\vec{\mathbf{e_i}})) = 0? \text{ and } L(\vec{\mathbf{e_i}}) = \lim_{t \to 0} \frac{1}{t} (\mathbf{f}(\mathbf{a} + t\vec{\mathbf{e_i}}) - \mathbf{f}(\mathbf{a}))$$

What is hard is proving that f is differentiable – that L exists – since that requires evaluating a limit where $\vec{\mathbf{h}} \to \vec{\mathbf{0}}$. Such a limit exists only if every sequence $\vec{\mathbf{h}}_1, \vec{\mathbf{h}}_2, \cdots$ that converges to $\vec{\mathbf{0}}$ leads to the conclusion that

$$\lim \frac{1}{|\vec{\mathbf{h}}_n|} \mathbf{r}(\vec{\mathbf{h}}_n) = 0$$

Mere existence of partial derivatives of $\mathbf{f}$ at $\mathbf{a}$ does not guarantee that $\mathbf{f}$ is differentiable at $\mathbf{a}$. Eventually we will prove (proof 11.2) that f is differentiable at $\mathbf{a}$ if all its partial derivatives are *continuous* there.

3. Proving differentiability and calculating derivatives

   In every case $\mathbf{f}$ is a function from U to $\mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$.

   - $\mathbf{f}$ is constant: $\mathbf{f} = \mathbf{c}$. Then $[\mathbf{Df}(\mathbf{a})]$ is the zero linear transformation, since

   $$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{f}(\mathbf{a}+\vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})]\vec{\mathbf{h}}) = \lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{c} - \mathbf{c} - \vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

   - $\mathbf{f}$ is affine: a constant plus a linear function, $\mathbf{f} = \mathbf{c} + L$. $[\mathbf{Df}(\mathbf{a})] = L$, since

   $$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{f}(\mathbf{a}+\vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - [\mathbf{Df}(\mathbf{a})]\vec{\mathbf{h}}) = \lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{c}+L(\mathbf{a}+\vec{\mathbf{h}}) - (\mathbf{c}+L(\mathbf{a})) - L(\vec{\mathbf{h}})) = 0.$$

   $\mathbf{f}$ has differentiable components: if $\mathbf{f} = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$: then $\mathbf{Df}(\mathbf{a}) = \begin{bmatrix} \mathbf{D}f_1(\mathbf{a}) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{D}f_n(\mathbf{a}) \end{bmatrix}$

   .

   - $\mathbf{f} + \mathbf{g}$ is the sum of two functions $\mathbf{f}$ and $\mathbf{g}$, both differentiable at $\mathbf{a}$. The derivative of $\mathbf{f} + \mathbf{g}$ is the sum of the derivatives of $\mathbf{f}$ and $\mathbf{g}$. (proof on next page)

4. Derivative of a sum

   f and **g** are differentiable at **a**. The obvious guess is that the derivative of **f** + **g** is the sum of their derivatives.

   Since **f** and **g** are differentiable at **a**, we know that

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{f}(\mathbf{a}+\vec{\mathbf{h}}) - \mathbf{f}(\mathbf{a}) - ([\mathbf{D}\mathbf{f}(\mathbf{a})]\vec{\mathbf{h}}) = 0.$$

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}(\mathbf{g}(\mathbf{a}+\vec{\mathbf{h}}) - \mathbf{g}(\mathbf{a}) - ([\mathbf{D}\mathbf{g}(\mathbf{a})]\vec{\mathbf{h}}) = 0.$$

   To prove the obvious guess, show that

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{1}{|\vec{\mathbf{h}}|}((\mathbf{f}+\mathbf{g})(\mathbf{a}+\vec{\mathbf{h}}) - (\mathbf{f}+\mathbf{g})(\mathbf{a}) - ([\mathbf{D}\mathbf{f}(\mathbf{a})] + [\mathbf{D}\mathbf{g}(\mathbf{a})])\vec{\mathbf{h}}) = 0.$$

5. Product rule (your proof 11.1):

   Now comes something harder: the product rule for two scalar-valued functions f and g. It is easy to guess what the derivative of fg must be, since in single variable calculus, $(fg)' = fg' + gf'$.

Product rule: $[\mathbf{D}(fg)(\mathbf{a})] = f(\mathbf{a})[\mathbf{D}g(\mathbf{a})] + g(\mathbf{a})[\mathbf{D}f(\mathbf{a})]$.

- Step 1: Write the "remainder" $r(\vec{\mathbf{h}})$ that must have the property

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{r(\vec{\mathbf{h}})}{|\vec{\mathbf{h}}|} = 0.$$

  $r(\vec{\mathbf{h}}) = f(\mathbf{a}+\vec{\mathbf{h}})g(\mathbf{a}+\vec{\mathbf{h}}) - f(\mathbf{a})g(\mathbf{a}) - f(\mathbf{a})[\mathbf{D}g(\mathbf{a})]\vec{\mathbf{h}} - g(\mathbf{a})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}$

- Step 2 – a trick that must be memorized: Subtract and add $f(\mathbf{a})g(\mathbf{a}+\vec{\mathbf{h}})$, and subtract and add $g(\mathbf{a}+\vec{\mathbf{h}})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}$.
  $r(\vec{\mathbf{h}}) = f(\mathbf{a}+\vec{\mathbf{h}})g(\mathbf{a}+\vec{\mathbf{h}}) - f(\mathbf{a})g(\mathbf{a}+\vec{\mathbf{h}}) + f(\mathbf{a})g(\mathbf{a}+\vec{\mathbf{h}}) - f(\mathbf{a})g(\mathbf{a})$
  $- f(\mathbf{a})[\mathbf{D}g(\mathbf{a})]\vec{\mathbf{h}} - g(\mathbf{a}+\vec{\mathbf{h}})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}} + g(\mathbf{a}+\vec{\mathbf{h}})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}} - g(\mathbf{a})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}$.

- Step 3: split into three terms, one involving the remainder for f, one involving the remainder for g, and one involving $[\mathbf{D}f(\mathbf{a})]$.
  $r(\vec{\mathbf{h}}) = r_1(\vec{\mathbf{h}}) + r_2(\vec{\mathbf{h}}) + r_3(\vec{\mathbf{h}})$, where
  $r_1(\vec{\mathbf{h}}) = f(\mathbf{a}+\vec{\mathbf{h}})g(\mathbf{a}+\vec{\mathbf{h}}) - f(\mathbf{a})g(\mathbf{a}+\vec{\mathbf{h}}) - g(\mathbf{a}+\vec{\mathbf{h}})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}$
  $r_2(\vec{\mathbf{h}}) = f(\mathbf{a})g(\mathbf{a}+\vec{\mathbf{h}}) - f(\mathbf{a})g(\mathbf{a}) - f(\mathbf{a})[\mathbf{D}g(\mathbf{a})]\vec{\mathbf{h}}$.
  $r_3(\vec{\mathbf{h}}) = g(\mathbf{a}+\mathbf{h})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}} - g(\mathbf{a})[\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}$.

- Step 4: Divide each term by $|\vec{\mathbf{h}}|$, and use the differentiability of f and g to prove that the limit $\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{r(\vec{\mathbf{h}})}{|\vec{\mathbf{h}}|}$ is zero. For each term you have the product of two factors: one approaches zero, while the other is bounded.
  $r_1(\vec{\mathbf{h}}) = (f(\mathbf{a}+\vec{\mathbf{h}}) - f(\mathbf{a}) - [\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}})g(\mathbf{a}+\vec{\mathbf{h}})$.

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{r_1(\vec{\mathbf{h}})}{|\vec{\mathbf{h}}|} = 0,$$

  since the first factor over $|\vec{\mathbf{h}}|$ goes to zero and the second is bounded.
  $r_2(\vec{\mathbf{h}}) = f(\mathbf{a})(g(\mathbf{a}+\vec{\mathbf{h}}) - g(\mathbf{a}) - [\mathbf{D}g(\mathbf{a})]\vec{\mathbf{h}})$.

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{r_2(\vec{\mathbf{h}})}{|\vec{\mathbf{h}}|} = 0,$$

  since the second factor over $|\vec{\mathbf{h}}|$ goes to zero and the first is constant.
  $r_3(\vec{\mathbf{h}}) = [g(\mathbf{a}+\vec{\mathbf{h}}) - g(\mathbf{a})][\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}$.

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{r_3(\vec{\mathbf{h}})}{|\vec{\mathbf{h}}|} = 0,$$

  since the first factor goes to zero by continuity and the second factor over $|\vec{\mathbf{h}}|$ is bounded.

6. (Proof 11.1)
   Let $U \subset \mathbb{R}^n$ be an open set, and let f and g be functions from U to $\mathbb{R}$. Prove that if f and g are differentiable at $\mathbf{a}$ then so is fg, and that

$$[\mathbf{D}(fg)(\mathbf{a})] = f(\mathbf{a})[\mathbf{D}g(\mathbf{a})] + g(\mathbf{a})[\mathbf{D}f(\mathbf{a})].$$

   (Simpler than in Hubbard because both f and g are scalar-valued functions)

7. Chain rule in $\mathbb{R}^n$ – not a proof, but still pretty convincing

The chain rule for differentiating composition of functions in general is as simple as you could hope for on the basis of single-variable calculus. Remember the single-variable version:

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

This can be rewritten

$$[D(f \circ g)(a)] = [Df(g(a))][Dg(a)],$$

where the square brackets convert the old-style derivatives (numbers) into 1×1 Jacobian matrices.

This says that the derivative of the composition of f and g is the composition of the linear function "multiply by $g'(a)$" and the linear function "multiply by $f'(g(a))$" Notice that f has to be differentiable at $g(a)$.

Here is the generalization:

$U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets, and $\mathbf{a}$ is a point in U at which we want to evaluate a derivative.

$\mathbf{g} : U \to V$ is differentiable at $\mathbf{a}$, and $[\mathbf{Dg}(\mathbf{a})]$ is a $m \times n$ Jacobian matrix.

$\mathbf{f} : V \to \mathbb{R}^p$ is differentiable at $\mathbf{g}(\mathbf{a})$, and $[\mathbf{Df}(\mathbf{g}(\mathbf{a}))]$ is a $p \times m$ Jacobian matrix.

The chain rule states that $[\mathbf{D}(\mathbf{f} \circ \mathbf{g})(\mathbf{a}))] = [\mathbf{Df}(\mathbf{g}(\mathbf{a}))] \circ [\mathbf{Dg}(\mathbf{a})]$.

Draw a diagram to illustrate what happens in the case $n = m = p = 2$ when you use derivatives to find a linear approximation to
$(\mathbf{f} \circ \mathbf{g})(\mathbf{a} + \tilde{\mathbf{h}}) - (\mathbf{f} \circ \mathbf{g})(\mathbf{a})$.
This approximation can be done in a single step or in two steps.

8. Two easy chain rule examples

(a) $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ maps time into the position of a particle moving around the unit circle:

$$\mathbf{g}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

$f : \mathbb{R}^2 \to \mathbb{R}$ maps a point into the temperature at that point.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2$$

The composition $f \circ \mathbf{g}$ maps time directly into temperature .
Confirm that $[D(f \circ g)(t)] = [\mathbf{D}f(\mathbf{g}(t))] \circ [\mathbf{D}\mathbf{g}(t)]$.

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be any differentiable function. You can make a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is constant on any circle centered at the origin by forming the composition $f\begin{pmatrix} x \\ y \end{pmatrix} = \phi(x^2 + y^2)$.
Show that f satisfies the partial differential equation $yD_1f - xD_2f = 0$.

9. Connection between Jacobian matrix and derivative

- If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is defined on an open set $U \in \mathbb{R}^n$, and

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{x}) \\ ... \\ f_m(\mathbf{x}) \end{pmatrix}$$

  the Jacobian matrix $[\mathbf{Jf}(\mathbf{x})]$ is made up of all the partial derivatives of $\mathbf{f}$:

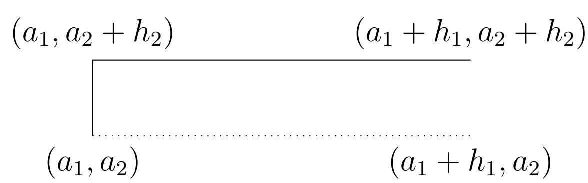
$$[\mathbf{Jf}(\mathbf{a})] = \begin{bmatrix} D_1 f_1(\mathbf{a})....D_n f_1(\mathbf{a}) \\ ... \\ D_1 f_m(\mathbf{a})....D_n f_m(\mathbf{a}) \end{bmatrix}$$

- We can invent pathological cases where the Jacobian matrix of f exists (because all the partial derivatives exist), but the function f is not differentiable. In such a case, using the formula

$$\nabla_{\vec{\mathbf{v}}} f(\mathbf{a}) = [\mathbf{J}f(\mathbf{a})]\vec{\mathbf{v}}$$

  generally gives the wrong answer for the directional derivative! You are trying to use a linear approximation where none exists.

- Using the Jacobian matrix of partial derivatives to get a good affine approximation for $f(\mathbf{a} + \vec{\mathbf{h}})$ is tantamount to assuming that you can reach the point $\mathbf{a} + \vec{\mathbf{h}}$ by moving along lines that are parallel to the coordinate axes and that the change in the function value along the solid horizontal line is well approximated by the change along the dotted horizontal line. With the aid of the mean value theorem, you can show that this is the case if (proof 11.2) the partial derivatives of f at $\mathbf{a}$ are continuous.



10. Jacobian matrix for a parametrization function gives a good affine approximation

Here is the function that converts the latitude u and longitude v of a point on the unit sphere to the Cartesian coordinates of that point.

$$f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}$$

Work out the Cartesian coordinates of the point with $\sin u = \frac{3}{5}$ (37 degrees North latitude) and $\sin v = 1$(90 degrees East longitude), and calculate the Jacobian matrix at that point. Then find the best affine approximation to the Cartesian coordinates of the nearby point where u is 0.01 radians less (going south) and v is 0.02 radians greater (going east).

11. A non-differentiable function

Consider a surface where the height z is given by the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{3x^2y - y^3}{x^2 + y^2}; f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

This function is not differentiable at the origin, and so you cannot calculate its directional derivatives there by using the Jacobian matrix!

(a) Along the first standard basis vector, the directional derivative at the origin is zero. Find two vectors along other directions that also have this property.

(b) Along the second standard basis vector, the directional derivative at the origin is -1.
Find two vectors along other directions that also have this property. (This surface is sometimes called a "monkey saddle," because a monkey could sit comfortably on it with its two legs and its tail placed along these three downward-sloping directions.)

(c) Calculate the directional derivative along an arbitrary unit vector $\vec{\mathbf{e}}_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Using the trig identity $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$, quickly rederive the special cases of parts (a) and (b).

(d) Using the definition of the derivative, give a convincing argument that this function is not differentiable at the origin.

12. The mean-value theorem in $\mathbb{R}^n$

For functions of one variable, this is an old friend.

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in [a, b]$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

The generalization uses the segment from $\mathbf{a}$ to $\mathbf{b}$ like the closed interval $[a, b]$.

The function f (takes values in $\mathbb{R}$) must be differentiable on an open set U that includes this entire segment.

The conclusion is that

$$f(\mathbf{b}) - f(\mathbf{a}) = [\mathbf{D}f(\mathbf{c})](\overrightarrow{\mathbf{b} - \mathbf{a}}).$$

The proof (Hubbard p. 148) is easy. Define a function $\mathbf{h}(t)$ that maps the interval $0 \le t \le 1$ uniformly into the segment from $\mathbf{a}$ to $\mathbf{b}$. The formula is

$$\mathbf{h}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

Now $g = f \circ \mathbf{h}$ satisfies all the hypotheses of the single-variable mean-value theorem. So there exists t_0 in (0,1) for which

$g(1) - g(0) = g'(t_0)(1 - 0)$.

By the chain rule, $g'(t_0) = [\mathbf{D}f(\mathbf{h}(t_0))]D\mathbf{h}(t_0)$

Set $\mathbf{c} = \mathbf{h}(t_0)$ and this becomes

$$f(\mathbf{b}) - f(\mathbf{a}) = [\mathbf{D}f(\mathbf{c})](\overrightarrow{\mathbf{b} - \mathbf{a}}).$$

If points $\mathbf{b}$ and $\mathbf{a}$ differ only in their ith component, so that $\mathbf{b} = \mathbf{a} + \vec{\mathbf{e}}_i$, then

$$f(\mathbf{a} + \vec{\mathbf{e}}_i) - f(\mathbf{a}) = D_i f(\mathbf{a} + t_0\vec{\mathbf{e}}_i), 0 < t_0 < 1.$$

13. (Proof 11.2) Using the mean value theorem, prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ that are continuous at $\mathbf{a}$, it is differentiable at $\mathbf{a}$ and its derivative is the Jacobian matrix $\begin{bmatrix} D_1 f(\mathbf{a}) & D_2 f(\mathbf{a}) \end{bmatrix}$.

14. Derivative of a function of a matrix (Example 1.7.17 in Hubbard):

A matrix is also a vector. When we square an $n \times n$ matrix A, the entries of $S(A) = A^2$ are functions of all the entries of A. If we change A by adding to it a matrix H of small length, we will make a change in the function value A^2 that is a linear function of H plus a small "remainder."

We could in principle represent A by a column vector with n^2 components and the derivative of S by a very large matrix, but it is more efficient to leave H in matrix form and use matrix multiplication to find the effect of the derivative an a small increment matrix H. The derivative is still a linear function, but it is represented by matrix multiplication in a different way.

(a) Using the definition of the derivative, show that the linear function that we want is $DS(H) = AH + HA$.

(b) Confirm that DS is a linear function of H

(c) Check that $DS(H)$ is a good approximation to $S(A+H) - S(A)$ for the following simple case, where the matrices A and H do not commute.
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $H = \begin{bmatrix} 0 & h \\ k & 0 \end{bmatrix}$

15. Derivative of the matrix-inverse function

    Define the function $T(A) = A^{-1}$. We expect that $T(A+H) - T(A)$ can be well approximated by an expression that is linear in H.

    Proof strategy: use a geometric series

    If we were dealing with numbers, we could write

$$\frac{1}{a+h} = \frac{1}{a}\frac{1}{1+\frac{h}{a}} = \frac{1}{a}(1 - \frac{h}{a} + \frac{h^2}{a^2} - ...) =$$

    This approach works with matrices, too, but we must be careful with the order of factors and remember that $(BC)^{-1} = C^{-1}B^{-1}$.

    - Prove that $(A+H)^{-1} = (I + A^{-1}H)^{-1}A^{-1}$.
    - Expand in a geometric series.
    - Evaluate $T(A+H) - T(A)$ and identify the term that is linear in H. Now we have our guess for the derivative.

- The "remainder" is $T(A+H) - T(A) + A^{-1}HA^{-1}$, and we have found that
  $T(A+H) = (A+H)^{-1} =$
  $A^{-1} - A^{-1}HA^{-1} + A^{-1}HA^{-1}HA^{-1} - A^{-1}HA^{-1}HA^{-1}HA^{-1} + \cdots.$

  Get a formula for this remainder that includes two factors of H. Then take its length. Use two strategies:

  Length of product $\leq$ product of lengths.

  Length of sum $\leq$ sum of lengths (generalized triangle inequality.)

- Prove that

$$\lim_{H \to 0} \frac{|Remainder|}{|H|} = 0.$$

16. Chain rule for functions of matrices

    We have shown that the derivative of the squaring function $S(A) = A^2$ is $DS(H) = AH + HA$

    We also showed that for $T(A) = A^{-1}$, the derivative is $DT(H) = -A^{-1}HA^{-1}$

    Now the function $U(A) = A^{-2}$ can be expressed as the composition $U = S \circ T$.

    Find the derivative $DU(H)$ by using the chain rule.

    The chain rule says "the derivative of a composition is the composition of the derivatives," even in a case like this where composition is not represented by matrix multiplication.

17. Newton's method

(a) One variable: Function f is differentiable. You are trying to solve the equation $f(x) = 0$, and you have found a value a_0, close to the desired x, for which $f(a_0)$ is small. Derive the formula $a_1 = a_0 - f(a_0)/f'(a_0)$ for an improved estimate.

(b) Use Newton's method to find an approximate value for the cube root of 8.1.

(c) n variables: U is an open subset of $\mathbb{R}^n$, and function $\vec{\mathbf{f}}(\mathbf{x}) : U \to \mathbb{R}^n$ is differentiable. You are trying to solve the equation $\vec{\mathbf{f}}(\mathbf{x}) = \vec{\mathbf{0}}$, and you have found a value $\mathbf{a_0}$, close to the desired $\mathbf{x}$, for which $\vec{\mathbf{f}}(\mathbf{a_0})$ is small. Derive the formula

$$\mathbf{a}_1 = \mathbf{a}_0 - [\mathbf{D}\tilde{\mathbf{f}}(\mathbf{a_0})]^{-1}\vec{\mathbf{f}}(\mathbf{a}_0).$$

for an improved estimate.

18. Newton's method – an example with two variables
We want an approximate solution to the equations

$\log x + \log y = 3$

$x^2 - y = 1$

$$\text{i.e. } f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \log x + \log y - 3 \\ x^2 - y - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Knowing that $\log 3 \approx 1.1$, show that $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$ is an approximate solution to this equation, then use Newton's method to improve the approximation.
Here is a check:

$\log 2.81 + \log 6.87 = 2.98$

$2.81^2 - 6.87 = 1.02$

19. Derivative of inverse function

Suppose that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. Choose a point $\mathbf{x}_0$ where the derivative $[\mathbf{Df}(\mathbf{x}_0)]$ is an invertible matrix. Set $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Let $\mathbf{g}$ be the differentiable local inverse function $\mathbf{g} = \mathbf{f}^{-1}$ such that

$\mathbf{g}(\mathbf{y}_0) = \mathbf{x}_0$ and $f(\mathbf{g}(\mathbf{y})) = \mathbf{y}$ if $\mathbf{y}$ is close enough to $\mathbf{y}_0$.

Prove that $[\mathbf{Dg}(\mathbf{y}_0)] = [\mathbf{Df}(\mathbf{x}_0)]^{-1}$

20. An economic example of the inverse-function theorem:

Your model: Providing x in health benefits and y in educational benefits leads to happiness H and cost C according the the equation

$$\begin{pmatrix} H \\ C \end{pmatrix} = \mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x^{0.5}y \\ x^{1.5} + y^{0.5} \end{pmatrix}.$$

Currently, $x = 4, y = 9, H = 22, C = 11$. Your budget is cut, and you are told to adjust x and y to reduce C to 10 and H to 19. Find an approximate solution by using the inverse-function theorem.

We cannot find formulas for the inverse function $\mathbf{g}\begin{pmatrix} H \\ C \end{pmatrix}$ that would solve the problem exactly, but we can calculate the derivative of $\mathbf{g}$.

(a) Check that$[\mathbf{Df}] = \begin{bmatrix} 1 + \frac{y}{2\sqrt{x}} & \sqrt{x} \\ \frac{3}{2}\sqrt{x} & \frac{1}{2\sqrt{y}} \end{bmatrix} = \begin{bmatrix} \frac{13}{4} & 2 \\ 3 & \frac{1}{6} \end{bmatrix}$ is invertible.

(b) Use the derivative $[\mathbf{Dg}] = \begin{bmatrix} -0.03 & 0.36 \\ 0.55 & -0.6 \end{bmatrix}$ to approximate $\mathbf{g}\begin{pmatrix} 19 \\ 10 \end{pmatrix}$

3 Group Problems

1. Chain rule

   (a) Chain rule for matrix functions

   On smple problem 4, we obtained the differentiation formula for $U(A) = A^{-2}$ by writing $U = S \circ T$ with $S(A) = A^2, T(A) = A^{-1}$. Prove the same formula from the chain rule in a different way, by writing $U = T \circ S$. You may reuse the formulas for the derivatives of S and T:

   If $S(A) = A^2$ then $[DS(A)](H) = AH + HA$.

   If $T(A) = A^{-1}$ then $[DT(A)](H) = -A^{-1}HA^{-1}$.

   (b) Chain rule with 2×2 matrices

   Start with a pair of polar coordinates $\begin{pmatrix} r \\ \theta \end{pmatrix}$.

   Function $\mathbf{g}$ converts them to Cartesian $\begin{pmatrix} x \\ y \end{pmatrix}$.

   Function $\mathbf{f}$ then converts $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$.

   Confirm that $[\mathbf{D}(\mathbf{f} \circ \mathbf{g})(\begin{pmatrix} r \\ \theta \end{pmatrix}))] = [\mathbf{Df}(\mathbf{g}\begin{pmatrix} r \\ \theta \end{pmatrix}))] \circ [\mathbf{Dg}\begin{pmatrix} r \\ \theta \end{pmatrix}]$

2. Issues of differentiability

   (a) Suppose that A is a matrix and S is the cubing function given by the formula $S(A) = A^3$. Prove that S is differentiable and that its derivative is the linear function of the matrix H given by the formula $[DS(A)](H) = A^2H + AHA + HA^2$.

       The proof consists in showing that the length of the "remainder" goes to zero faster than the length of the matrix H.

   (b) A continuous but non-differentiable function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2y}{x^2+y^2}, f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

      i. Show that both partial derivatives vanish at the origin, so that the Jacobian matrix at the origin is the zero matrix $[0\ 0]$, but that the directional derivative along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not zero. How does this calculation show that the function is not differentiable at the origin?

      ii. For all points except the origin, the partial derivatives are given by the formulas

$$D_1f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2xy^3}{(x^2+y^2)^2}, D_2f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^4-x^2y^2}{(x^2+y^2)^2}$$

      Construct a "bad sequence" of points approaching the origin to show that D_1f is discontinuous at the origin.

3. Inverse functions and Newton's method

(a) An approximate solution to the equations

$$x^3 + y^2 - xy = 1.08$$

$$x^2y + y^2 = 2.04$$

is $x_0 = 1$, $y_0 = 1$.

Use one step of Newton's method to improve this approximation.

(b) You are in charge of building the parking lots for a new airport. You have ordered from amazon.com enough asphalt to pave 1 square kilometer, plus 5.6 kilometers of chain-link fencing. Your plan is to build two square, fenced lots. The short-term lot is a square of side x=0.6 kilometers; the long-term lot is a square of side y=0.8 kilometers. The amount of asphalt A and the amount C of chain-link fencing required are then specified by the function

$$\begin{pmatrix} A \\ C \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ 4x + 4y \end{pmatrix},$$

Alas, Amazon makes a small shipping error. They deliver enough asphalt to pave 1.03 square kilometers but only 5.4 kilometers of fence.

i. Use the inverse-function theorem to find approximate new values for x and y that use exactly what was shipped to you.
In this simple case you can check your answer by solving algebraically for x and y.

ii. Find a case where $A = 1$ but the value of C is such that this approach will fail because $[DF]$ is not onto. (This case corresponds to the maximum amount of fencing.)

4. Problems to be solved using R (if your group has experience with R, solve one of these as your third problem.)

   Both problems can by done by modifying R script 3.3B. Since Newton's method is the only generally applicable technique for solving nonlinear equations, it is really useful to know how to do it on a computer.

   (a) Saving Delos

   The ancient citizens of Delos, threatened with a plague, consulted the oracle of Delphi, who told them to construct a new cubical altar to Apollo whose volume was double the size of the original cubical altar. (For details, look up "Doubling the cube" on Wikipedia.)

   If the side of the original altar was 1, the side of the new altar had to be the real solution to $f(x) = x^3 - 2 = 0$.

   Numerous solutions to this problem have been invented. One uses a "marked ruler" or "neusis"; another uses origami.

   Your job is to use multiple iterations of Newton's method to find an approximate solution for which $x^3 - 2$ is less than 10^{-8} in magnitude.

   (b) An approximate solution to the system of nonlinear equations

$$x + y^2 + z^3 = 9$$
$$xy + xz + yz = 12$$
$$xyz = 7$$

   is $x = 3, y = 2, z = 1$.

   Use two iterations of Newton's method to find a good approximate solution to these equations.

4 Homework

1. (similar to group problem 1a)

   We know the derivatives of the matrix-squaring function S and the matrix-inversion function T:

   If $S(A) = A^2$ then $[DS(A)](H) = AH + HA$.

   If $T(A) = A^{-1}$ then $[DT(A)](H) = -A^{-1}HA^{-1}$.

   (a) Use the chain rule to find a formula for the derivative of the function $U(a) = A^4$.

   (b) Use the chain rule to find a formula for the derivative of the function $W(a) = A^{-4}$.

2. (a) Hubbard, Exercise 1.7.21 (derivative of the determinant function). This is really easy if you work directly from the definition of the derivative.

   (b) Generalize this result to the 3×3 case. Hint: consider a matrix whose columns are $\vec{\mathbf{e}}_1 + h\vec{\mathbf{a}}_1$, $\vec{\mathbf{e}}_2 + h\vec{\mathbf{a}}_2$, $\vec{\mathbf{e}}_3 + h\vec{\mathbf{a}}_3$, and use the definition of the determinant as a triple product.

3. Hubbard, Exercise 1.8.6, part (b) only. In the case where **f** and **g** are functions of time t, this formula finds frequent use in physics. You can either do the proof as suggested in part (a) or model your proof on the one for the dot product on page 143.

4. (similar to the second example on page 14)

   Hubbard, Exercise 1.8.9. The equation that you prove can be called a "first-order partial differential equation."

5. (similar to group problem 2b)

As a summer intern, you are given the job of reconciling the Democratic and Republican proposals for tax reform. Both parties agree on the following model:

- x is the change in the tax rate for the middle class.
- y is the change in the tax rate for the well-off.
- The net impact on revenue is given by the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x(x^2 - y^2)}{x^2 + y^2}, \; f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

The Republican proposal is $y = -x$, while the Democratic proposal is $y = x$.

(a) Show that f is continuous at the origin.

(b) Show that both proposals are revenue neutral by calculating two appropriate directional derivatives. You will have to use the definition of the directional derivative, not the Jacobian matrix.

(c) At the request of the White House, you investigate a 50-50 mix of the two proposals, the compromise case where $y = 0$, and you discover that it is not revenue neutral! Confirm this surprising conclusion by showing that the directional derivatives at the origin cannot be given by a linear function; i.e. that f is not differentiable.

(d) Your final task is to explain the issue in terms that legislators can understand: the function is not differentiable because its partial derivatives are not continuous. Demonstrate that one of the partial derivatives of f is discontinuous at the origin. ($D_2 f$ is less messy.)

6. Chain rule: an example with 2×2 matrices
A similar example with a 3×3 matrix is on page 151 of Hubbard.

The function
$\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x+y) \\ \sqrt{xy} \end{pmatrix}$ was invented by Gauss about 200 years ago to deal with integrals of the form

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(t^2+x^2)(t^2+y^2)}}.$$

It was revived in the late 20th century as the basis of the AGM (arithmetic-geometric mean) method for calculating π. You can get 1 million digits with a dozen or so iterations.

The function is meant to be composed with itself; so it will be appropriate to compute the derivative of $\mathbf{f} \circ \mathbf{f}$ by the chain rule.

(a) $\mathbf{f}$ is differentiable whenever x and y are positive; so its derivative is given by its Jacobian matrix. Calculate this matrix.

We choose to evaluate the derivative of $\mathbf{f} \circ \mathbf{f}$ at the point $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$.

Conveniently, $\mathbf{f}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. The chain rule says that

$$[\mathbf{D}(\mathbf{f} \circ \mathbf{f})]\begin{pmatrix} 8 \\ 2 \end{pmatrix} = [\mathbf{Df}\begin{pmatrix} 5 \\ 4 \end{pmatrix}][\mathbf{Df}\begin{pmatrix} 8 \\ 2 \end{pmatrix}].$$

Evaluate the two numerical Jacobian matrices. Because the derivative of f is evaluated at two different points, they will not be the same.

(b) Write the formula for $\mathbf{f} \circ \mathbf{f}$, compute and evaluate the lower left-hand entry in its Jacobian matrix, and check that it agrees with the value given by the chain rule.

7. (a) Hubbard, problem 2.10.2. Make a sketch to show how this mapping defines an alternative coordinate system for the plane, in which a point is defined by the intersection of two hyperbolas.

   (b) The point $x = 3, y = 2$ is specified in this new coordinate system by the coordinates $u = 6, v = 5$. Use the derivative of the inverse function to find approximate values of x and y for a nearby point where $u = 6.5, v = 4.5$. (This is essentially one iteration of Newton's method.)

   (c) Find h such that the point $u = 6 + h, v = 5.1$ has nearly the same x-coordinate as $u = 6, v = 5$.

   (d) Find k such that the point $x = 3 + k, y = 2.1$ has nearly the same u-coordinate as $x = 3, y = 2$.

   (e) For this mapping, you can actually find a formula for the inverse function that works in the region of the plane where $x, y, u,$ and v are all positive. Find the rather messy formulas for x and y as functions of u and v, and use them to answer the earlier questions. Once you calculate the Jacobian matrix and plug in appropriate numerical values, you will be back on familiar ground.

8. The CEO of a chain of retail stores will get a big bonus if she hits her volume and profit targets for December exactly. Her microeconomics consultant, fresh out of Harvard, tells her that both her target figures are functions of two variables, investment x in Internet advertising and investment y in television advertising. The former attracts savvier customers and so tends to contribute to volume more than to profit.

   The function that determines volume V and profit P is

$$\begin{pmatrix} V \\ P \end{pmatrix} = \begin{pmatrix} x^{\frac{3}{4}}y^{\frac{1}{3}} + x \\ x^{\frac{1}{4}}y^{\frac{2}{3}} + y \end{pmatrix}.$$

   With $x = 16, y = 8, V = 32, P = 16$, our CEO figures she is set for a big bonus. Suddenly, the board of directors, feeling that Wall Street is looking as much for profit as for volume this year, changes her targets to $V = 24, P = 24$. She needs to modify x and y to meet these new targets.

   Near $V = 32, P = 16$, there is an inverse function such that $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{g}\begin{pmatrix} V \\ P \end{pmatrix}$. Find its derivative $[\mathbf{Dg}]$, and use the derivative to find values of x and y that are an approximate solution to the problem. Because the increments to V and P are large, you should not expect the approximate solution to be very good, but it will be better than doing nothing.

   Optional extra problems to be solved using R

9. In the preceding problem, use multiple iterations of Newton's method in R to find accurate values of x and y that meet the revised targets. Feel free to modify Script 3.3C.

10. (Related to group problem 4a)

    The quintic equation $x(x^2 - 1)(x^2 - 4) = 0$ clearly has five real roots that are all integers. So does the equation $x(x^2 - 1)(x^2 - 4) - 1 = 0$, but you have to find them numerically. Get all five roots using Newton's method, carrying out enough iterations to get an error of less than .001. Use R to do Newton's method and to check your answers. If you have R plot a graph, it will be easy to find an initial guess for each of the five roots.