Math 221 : Algebra notes Nov. 20

Alison Miller

1 Examples and some basic properties of induced representations

Last time we stated this definition:

Let $\rho : G \to GL(V)$ be a representation of G. Let $W \subset V$ be a subspace that is H-invariant; let $\theta : H \to GL(W)$ be the corresponding representation of H. For every $g \in G$, we have a subspace $\rho_g(W) \subset V$; this only depends on the left coset gH. So if σ is any left coset of H in G, we can define $W_\sigma = \rho_g(W)$ for any $g \in \sigma$.

Definition. We say that ρ is induced by θ if $V = \oplus_{\sigma \in G/H} V_\sigma$.

Now we give some examples.

Example. $\rho : G \to GL(V)$ is the regular representation with basis $\{e_g\}_{g \in G}$, and $W = \mathrm{span}(e_h)_{h \in H}$ is the regular representation of H. Then $W_\sigma = \mathrm{span}(e_g)_{g \in \sigma}$, and $V = \oplus_{\sigma \in G/H} W_\sigma$.

Example. $\rho : G \to GL(V)$ is the permutation representation on left cosets of H, with basis $\{e_\sigma\}_{\sigma \in G/H}$, and $W = \mathrm{span}(e_H)$, θ is the trivial representation of W. Then $W_\sigma = \mathrm{span}(e_\sigma)$ and again $V = \oplus_{\sigma \in G/H} W_\sigma$

Example. $G = D_n$, $\rho : G \to GL(V)$ is the 2-dimensional representation given by embedding G into $GL_2(\mathbb{C})$ as the symmetry group of a regular n-gon, $H = C_n$. Here we may take $W = \mathrm{span}(\begin{pmatrix} 1 \\ i \end{pmatrix})$. In this case, there are only two cosets, H and gH for any $g \notin H$. Clearly $W_H = W$, and to find W_{gH} we can choose g such that ρ_g is reflection through the x-axis, so $W_{gH} = \mathrm{span}(\rho_g(\begin{pmatrix} 1 \\ i \end{pmatrix})) = \mathrm{span}(\begin{pmatrix} 1 \\ -i \end{pmatrix})$. Clearly $V = W_H \oplus W_{gH}$.

Observations: if $\rho : G \to GL(V)$ is induced by $\theta : G \to GL(W)$, and W' is an H-invariant subspace of W, then $V' = \bigoplus_{\sigma \in G/H} W'_\sigma$ is G-invariant, and the representation V' of G is induced by the representation W' of H.

If V_1 is induced by W_1 and V_2 is induced by W_2, then $V_1 \oplus V_2$ is induced by $W_1 \oplus W_2$.

Using this, we can show that for any representation W of H there is some representation V of G which is induced by W.

First, we do this when W is irreducible. We know that the regular representation W_{reg} of H contains W as a summand in any irreducible decomposition. Hence we can choose an injection $W \hookrightarrow W_{\text{reg}}$ of H-representations and identify W with its image inside W_{reg}. Now, the regular representation V_{reg} of G is induced by W_{reg}, so by the first observation above, V_{reg} has a subspace V which is induced by W.

Now, let W be an arbitrary representation of H, and take an irreducible decomposition $W = \oplus_i W_i$. By the previous paragraph, there are representations V_i of G induced by W_i, and then by the second observation, $\oplus_i V_i$ is induced by $W = \oplus_i W_i$.

Although this works to show that V exists, it is not very canonical, in that it required taking a choice of embedding of each W_i into W_{reg}. A more canonical construction is given in your problem set.

2 Universal property of the induced representation

However, we'll now show that the induced representation V of G is determined up to canonical isomorphism by the representation of W. To do that, we'll show it has the following universal property:

Theorem 2.1. *If $\rho : G \to GL(V)$ is induced by $\theta : G \to GL(W)$, then for any other representation $\rho' : G \to GL(V')$ and any homomorphism $f : W \to V'$ of H-representations, there is a unique homomorphism $\tilde{f} : V \to V'$ of G-representations such that $\tilde{f}|_W = f$.*

Proof. We'll do uniqueness first, then existence:

Uniqueness: Since $V = \oplus_{\sigma \in G/H} W_\sigma$, to show that $\tilde{f}$ is uniquely determined, it's enough to show that $\tilde{f}|_{W_\sigma}$ is uniquely determined.

For any $\sigma \in G/H$, choose a coset representative $g \in \sigma$. Now, an arbitrary element of W_σ is of the form $\rho_g(w)$ for some $w \in W$. Because $\tilde{f}$ is a homomorphism of G-representations, we have

$$\tilde{f}(\rho_g(w)) = \rho'_g(\tilde{f}(w)) = \rho'_g(f(w))$$

since $\tilde{f}|_W = f$.

Hence the conditions imposed determine the values of $\tilde{f}|_{W_\sigma}$ for any $\sigma \in G/H$, hence determine $\tilde{f}$.

Existence: From the above, we get a formula for $\tilde{f}|_{W_\sigma}$ for each $\sigma \in G/H$, and so also for $\tilde{f}$. To check that this works we need to check two things: that the formula for $\tilde{f}|_{W_\sigma}$ does not depend on the choice of $g \in \sigma$, and that $\tilde{f} : V \to V'$ is indeed a homomorphism of G-representations.

□

Corollary 2.2. *If W is a representation of H, and V_1, V_2 are representations of G both induced by W, there is a unique isomorphism $V_1 \cong V_2$ which restricts to the identity on W.*

Proof. This is a standard universal property argument. Let $i_1 : W \to V_1$ and $i_2 : W \to V_2$ be the inclusion maps. Then our universal property gives us unique maps $\tilde{i}_1 : V_2 \to V_1$ and $\tilde{i}_2 : V_1 \to V_2$ such that $\tilde{i}_1 \circ i_2 = i_1$ and $\tilde{i}_2 \circ i_1 = i_2$. Then we argue as in the usual universal property argument that $\tilde{i}_1$ and $\tilde{i}_2$ are inverses. $\square$

Now a bit of notation.

Definition. If $H \subset G$, and W is a representation of H, we denote the representation induced by W (which we now know is determined up to unique isomorphism by $\operatorname{Ind}_H^G(V)$ or just $\operatorname{Ind} V$ if G and H are clear from context.

If $\rho : G \to GL(V)$ is a representation of V, we use the notation $\operatorname{Res}_H^G V$ for the restricted homomorphism $\rho|_H : H \to GL(V)$.

With this notation, we can restate our universal property as follows:

Proposition 2.3. *There is a natural identification*

$$\operatorname{Hom}_H(W, \operatorname{Res} V') \cong \operatorname{Hom}_G(\operatorname{Ind} W, V')$$

given by $f \mapsto \tilde{f}$ *and* $g|_W \leftarrow g$.