**Forces Change Momentum**

At this point, we’ve talked a great deal about *isolated* systems. As you know, the total momentum of any isolated system—no matter how complex—is constant:

For an isolated system, \( \frac{d}{dt}(\vec{p}_{\text{tot}}) = 0 \)

Although this fact about isolated systems is useful in those limited cases when we can treat a real system as approximately isolated, nearly all interesting systems are not isolated. Indeed, most real systems exhibit both internal and external interactions. As you recall, any system that participates in external interactions (interactions between an object inside the system and an object outside the system) is not isolated:

We now must consider how a system will behave if it is *not* isolated. Newton proposed that the time rate of change of the momentum of a system is equal to the sum of all the external forces on that system. That is, he proposed:

For a non-isolated system, \( \frac{d}{dt}(\vec{p}_{\text{tot}}) = \sum \vec{F}_{\text{ext}} \)

These forces, which change the momentum of a system, arise from the interactions between various objects. External forces arise from external interactions; internal forces arise from internal interactions. This proposal, that forces change momentum, is the heart of Newton’s system of mechanics and is indeed the original statement of Newton’s Second Law.

**Forces on a Single Object: Newton’s Second Law**

Usually in Newtonian mechanics we will choose our “system” to consist of a single object; any system of more that one object can be broken down into several simpler systems, each of which has only one object. For a single object, of course, \( \vec{p} = m\vec{v} \). (From our previous discussion, we know that the velocity \( \vec{v} \) in this expression is the velocity of the center of mass of the object.) Thus, if our system consists of a single object, we can write:
\[
\frac{dp}{dt} = \sum \vec{F}_{ext}
\]

You should read this equation as “The time rate of change of the momentum of an object is equal to the sum of all external forces acting on that object.” This expression is close to Newton’s original statement of the second law:

\[
Lex \ II: \ Mutationem \ motus \ proportionalem \ esse \ vi \ motrici \ impressae \ et \ fieri \ secundum \ lineam \ rectam \ qua \ vis \ illa \ imprimitur.
\]

\textbf{Law II: The rate of change of motion (momentum) of a body is proportional to the force acting on the body and is in the same direction.}

Although this equation may look unfamiliar, we can cast it in a familiar form if we turn it around and remember that we’re dealing with objects whose mass is essentially constant:

\[
\sum \vec{F}_{ext} = \frac{dp}{dt} = \frac{d}{dt} (m\vec{v}) = m\frac{d\vec{v}}{dt}
\]

This equation states that the sum of the external forces is equal to the mass multiplied by the time rate of change of the velocity. We can now define a new quantity, the \textbf{acceleration}, as the time rate of change of the velocity:

\textbf{Definition of acceleration:} \[ \vec{a} = \frac{d\vec{v}}{dt} \]

By defining acceleration, we arrive at the familiar form of Newton’s Second Law:

\[
\sum \vec{F}_{ext} = m\vec{a}
\]

Although this form of the Second Law (“\(F = ma\)”) is far more familiar than the others, many instructors have found that students achieve better comprehension of Newtonian mechanics if they think of the Second Law as “Forces change velocity” or “Forces change momentum” rather than “Forces cause acceleration.” Now that you have a good understanding of momentum, you should read this law as: if the \textit{momentum} of an object changes, that change must be the result of a \textit{nonzero net external force} on the object.

Just as our equations for the conservation of momentum were vector equations, these equations for Newton’s Second Law are also vector equations. In any Cartesian coordinate system, we can separate this single vector equation into three equations, one for each component:

\[
\sum F_x = \frac{dp_x}{dt}
\]
\[ \sum F_y = \frac{dp_y}{dt} \]
\[ \sum F_z = \frac{dp_z}{dt} \]

(We will, from now on, drop the subscript “ext,” but you must remember that only external forces can change the momentum of a system.)

The Second Law, which is the heart of Newtonian Mechanics, can be used in two ways: we can observe a change in the momentum of an object \( (d\dot{p}/dt) \) and deduce the external forces that must have produced that change in momentum, or we can start with a knowledge of the forces on an object and calculate how the momentum will change as a result.

Just as we must distinguish between the instantaneous velocity \( (d\vec{r}/dt) \) and the average velocity \( (\Delta\vec{r}/\Delta t) \), we can must distinguish between the instantaneous force and the average force.

We define the average force \( \langle \vec{F} \rangle \) over a finite period of time \( \Delta t \) as:

**Average Force:** \[ \sum \langle \vec{F} \rangle = \langle \sum \vec{F} \rangle = \frac{\Delta p}{\Delta t} \]

Note that there is no difference between the the sum of the average forces or the average of the sum of the forces; both expressions are mathematically equivalent.

Finally, we should consider the dimensions and SI units of force. Since momentum has dimensions of \([M][L][T]^{-1}\), and force is the change in momentum over time, we can deduce:

**force:** \( \vec{F} = [M][L][T]^{-2} \) with SI units of: \( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \)

The SI unit of the **newton** \( (N) \) is the special name given to the unit \( \text{kg} \cdot \text{m}/\text{s}^2 \). Compared to the macroscopic forces with which you are familiar (such as the forces exerted by your muscles), one newton is a relatively weak force. If you hold an apple in your hand, the force of the apple on your hand is about equal to one newton.

**Example: An Object in Free Fall**

Let’s consider a simple example. Suppose you have a 5-kg stone suspended by a rope from the Golden Gate Bridge. The stone is not currently moving, so its momentum is zero. You cut the rope, and the stone falls towards the water. You measure the velocity of the stone as it falls, and find that after 2 seconds the stone has a velocity of \( (-19.6 \, \text{m/s}) \hat{y} \) in a coordinate system in which the \( y \)-direction is “up.” What was the average force on the stone as it fell?
The initial momentum was \( \vec{p} = 0 \), because the stone was at rest. After 2 seconds, the final momentum was \( \vec{p} = mv \), or \( \vec{p} = (5 \text{ kg})(-19.6 \text{ m/s})\hat{y} = (-98 \text{ kg\cdot m/s})\hat{y} \). The total average force during that time was:

\[
\text{Average Force: } \sum \langle \vec{F} \rangle = \frac{\Delta \vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{t_f - t_i} = \frac{(-98 \text{ kg\cdot m/s})\hat{x} - 0}{2 \text{ s} - 0} = (-49 \text{ kg\cdot m/s}^2)\hat{x}
\]

So the average force on the 5-kg stone had a magnitude of 49 N and was directed “down,” toward the center of the Earth. What could be the origin of this force? Consider all the external interactions with the stone as it falls. It can only participate in two kinds of interactions: contact interactions and long-range interactions. The only possible contact interaction is an interaction with the air, and the air friction (or drag) on this small falling stone is relatively small. (Later in the course, you will learn how to calculate the air drag on moving objects.) The only long-range interaction is a gravitational interaction with the Earth. This gravitational interaction contributes a force on the stone that pulls it towards the Earth. The gravitational force on an object near the surface of the Earth points toward the center of the Earth and has a magnitude equal to:

\[
F_{\text{grav}} = mg
\]

where \( m \) is the mass of the object and \( g \) is the gravitational acceleration of roughly 9.8 m/s\(^2\). For a 5-kg stone, this force is 49 newtons. Since this gravitational force is essentially constant, the average force and the instantaneous force on the falling stone are equal.

You should learn to look closely at objects in your everyday life and ask the following questions:

- Is an object changing its momentum? (Remember, momentum is a vector quantity, so it could be changing its speed and/or its direction.)
- If its momentum is changing, what are the external interactions that could be changing its momentum? (Remember that all interactions are either contact interactions or long-range interactions.)
- What are the forces on the object that result from these interactions? The vector sum of all these forces must equal the time rate of change of the object’s momentum.

You will greatly increase your comprehension of Newtonian mechanics if you ask (and answer) these simple questions whenever you have a spare moment.
Interactions Between Two Objects: Newton’s Third Law

Let’s consider an isolated system of two objects in which the two objects are interacting in some way:

We know that the total momentum of this system must be constant, and that the total momentum is simply the sum of the momentum of object 1 plus the momentum of object 2. We apply conservation of momentum to the entire system:

\[
\frac{d}{dt}(\vec{p}_{\text{tot}}) = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0
\]

As we saw earlier, when we discussed the conservation of momentum, we can rearrange this expression to obtain:

\[
\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}
\]

Let’s take advantage of the fact that we can define the system in any way we choose, and consider the same physical situation, but now consider our system to consist only of object 1. From this point of view, the interaction between the two objects is an external interaction:

Now the time rate of change of the momentum of object 1 is equal to the force exerted by object 2 on object 1, which we will represent as \( \vec{F}_{2\rightarrow 1} \). (This force is the only force on object 1, since it is the only interaction in this particular situation.) We can write:

\[
\frac{d\vec{p}_1}{dt} = \vec{F}_{2\rightarrow 1}
\]
Alternatively, we could consider our system to consist only of object 2:

By the same argument, the time rate of change of the momentum of object 2 is equal to the force exerted by object 1 on object 2, which we will represent as $\vec{F}_{1\rightarrow 2}$. We can write:

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{1\rightarrow 2}$$

If we substitute these two equations into the expression we found above for the conservation of momentum of the whole system (of both objects), we conclude:

$$\vec{F}_{2\rightarrow 1} = -\vec{F}_{1\rightarrow 2}$$

That is, the force of object 2 on object 1 is equal in magnitude, and pointing in the opposite direction to the force of object 1 on object 2. This is a specific example of Newton’s Third Law, which is sometimes paraphrased as “For every action there is an equal and opposite reaction.”

Although we derived this law for the case of two objects in an isolated system, it is easy to show that this law holds in general for any number of objects. You must note that the pairs of opposite forces in the Third Law always act upon different objects. For instance, when you stand on the ground, there are two forces on your body: the long-range force of gravity on your body and the contact force of the ground on your body. These two forces are not related in any way by Newton’s Third Law, since they both act upon the same object (your body).

To understand the meaning of Newton’s Third Law, consider how you jump. When you jump, you use your feet to exert a force against the ground; we can denote the force of your feet on the ground as $\vec{F}_{\text{feet-G}}$. This force is directed “down.” By the Third Law, there must be an equal and opposite force of the ground on your feet, $\vec{F}_{G\rightarrow \text{feet}}$, that is directed “up.” This force is the one that propels your body upward: it is a force that acts on your body to change its momentum from zero to something pointing “up.” Such a force is often called a reaction force; it is the reaction force of the ground on our bodies that allows us to walk, run, and jump. Look around at the forces you see in your everyday life, and you will see numerous examples of Newton’s Third Law at work: these “reaction forces” are everywhere!
The Total Change in Momentum: The Impulse

There are many physical situations in which a relatively large force acts over a relatively short period of time; these situations are often called collisions. Examples include a ball bouncing against a wall, a bug being crushed against a car windshield, or a molecule colliding with another molecule. Although it may be impossible to calculate the exact forces during a collision, it is possible to determine the total average force by examining the change in momentum during the collision. Recall that the average force is given by:

$$\text{Average Force: } \sum \langle \vec{F} \rangle = \langle \sum \vec{F} \rangle = \frac{\Delta \vec{p}}{\Delta t}$$

Thus, if an object changes its momentum from $\vec{p}_i$ to $\vec{p}_f$ during some time $\Delta t$, the total average force on the object is given by $\Delta \vec{p} / \Delta t$. In the case of collisions, the change in momentum $\Delta \vec{p}$ during the collision is called the impulse. Although any change in momentum could be called an impulse, that term is usually reserved for instances in which the momentum changes over a relatively brief period of time.

As an example, consider a 100-kg mass that is moving in the negative-$x$ direction with a constant speed of 50 m/s. You apply a force for a duration of one second in the positive-$x$ direction. After applying this force, the mass is now moving with a speed of 50 m/s in the positive-$x$ direction. The following diagram summarizes this scenario:

![Diagram showing a 100 kg mass moving in the negative x-direction with an initial velocity of 50 m/s. A force is applied for one second, changing the direction to positive x, with a final velocity of 50 m/s.](attachment:image.png)

(We assume here that there is no net force on the object before or after the one-second period in which we apply a force.) What was the average force that you applied to reverse the direction of this object?

The change in momentum, $\Delta \vec{p}$, is:

$$\Delta \vec{p} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i) = (100 \text{ kg})[(50 \text{ m/s}) - (-50 \text{ m/s})]\hat{x} = (10^4 \text{ kg} \cdot \text{m/s})\hat{x}$$
The total average force is therefore:

$$\sum \langle \bar{F} \rangle = \frac{\Delta \bar{p}}{\Delta t} = \frac{(10^4 \text{kg} \cdot \text{m/s})\hat{x}}{1 \text{ s}} = (10^4 \text{kg} \cdot \text{m/s}^2)\hat{x} = (10^4 \text{N})\hat{x}$$

or, stated in words, the average force is \(10^4\) newtons, in the positive-\(x\) direction. This general approach allows us to calculate the total average force during any collision (or indeed any process). For instance, we can use this strategy to estimate the average force on a person during an auto accident, or the average force on your leg bones during each step while you’re running, or the average force on a flea when it jumps. In many biological situations, when the detailed forces would be impossible to determine, this approach to estimating the average force is tremendously valuable.

We conclude this section by considering what we can learn when we do know the instantaneous force at any time during the collision. The total instantaneous force on an object is given by Newton’s Second Law:

$$\sum \bar{F} = \frac{d\bar{p}}{dt}$$

For simplicity, we’ll drop the “summation” sign in the following derivation; please keep in mind that the force under discussion is the total force acting on the object:

$$\bar{F} = \frac{d\bar{p}}{dt}$$

We can rearrange this expression to find the infinitesimal change in momentum produced by the total force \(\bar{F}\) acting over the infinitesimal time \(dt\):

$$d\bar{p} = \bar{F} \, dt$$

Now, we integrate that expression from some initial conditions (\(\bar{p}_i\) and \(t_i\)) to some final conditions (\(\bar{p}_f\) and \(t_f\)):

$$\int_{\bar{p}_i}^{\bar{p}_f} d\bar{p} = \int_{t_i}^{t_f} \bar{F} \, dt$$

The left-hand side is simply the change in momentum—the impulse—which is equal to \(\Delta \bar{p}\):

$$\bar{p}_f - \bar{p}_i = \Delta \bar{p} = \int_{t_i}^{t_f} \bar{F} \, dt$$

The right-hand side is an integral that can only be evaluated if we know the instantaneous force \(\bar{F}\) as a function of time during the collision. If the force is constant, then we can take it out of the integral, and we obtain:
\[
\Delta \vec{p} = \int_{t_i}^{t_f} \vec{F} \, dt = \vec{F} \Delta t
\]

If the force is constant, the instantaneous force is the same as the average force, so we have:

\[
\vec{F} = \frac{\Delta \vec{p}}{\Delta t}
\]

Let’s examine this case of a constant force. What happens in our example if we apply a constant force of \(10^4\) N for 1 second? A graph of the force as a function of time would look like this:

Integrating that constant force, and using the fact that the \(x\)-component of the momentum was initially \(-5000\) kg·m/s, we obtain the \(x\)-component of the momentum as a function of time:
You should see from these graphs that the force is the derivative of the momentum with respect to time or, conversely, that the momentum is the integral of the force with respect to time. These relationships emerge from Newton’s Second Law, \( \ddot{F} = d\dot{p}/dt \). As shown by the graphs on the previous page, if you apply a constant force to an object, the momentum changes at a constant rate (i.e. with a constant slope). The total impulse (the total change in momentum) is given by the area under the curve on the graph of the force as a function of time.

Now let’s consider a more realistic situation. Instead of instantly applying a constant force over a period of one second, you increase the force and then decrease it smoothly. Perhaps the force as a function of time looks something like this:

![Graph of a parabola](image)

This is a graph of a parabola; in this case, the equation for the x-component of the force (between the times \( t = 0 \) and \( t = 1 \) second) is:

\[
F = At - Bt^2 \quad \text{with} \quad A = 6 \times 10^4 \text{ N/s} \quad \text{and} \quad B = 6 \times 10^4 \text{ N/s}^2
\]

You should be able to confirm that this force is zero when \( t = 0 \); zero when \( t = 1 \) second; and equal to 15000 N when \( t = 0.5 \) seconds, as shown on the graph. We can find the change in momentum between time \( t_i = 0 \) and any time \( t \) (up to \( t_f = 1 \) second) by integrating:

\[
\dot{p} - \dot{p}_i = \Delta \dot{p} = \int_{t_i=0}^{t_f} \ddot{F} \, dt = \int_{t_i=0}^{t_f} (At - Bt^2) \, dt = \frac{1}{2}At^2 - \frac{1}{3}Bt^3
\]

This force gives the same total impulse as the previous case involving a constant force: the total impulse from time \( t_i = 0 \) to time \( t_f = 1 \) second is equal to \( 10^4 \) kg\cdot m/s.
We can graph this expression for the momentum as a function of time. To facilitate comparison, here is a graph of the force as a function of time (the same graph as on the previous page:

![Force vs Time Graph](attachment:force_graph.png)

Now, here is the graph of the $x$-component of the momentum as a function of time:

![Momentum vs Time Graph](attachment:momentum_graph.png)

Once again, you should see that the force is the derivative of the momentum with respect to time, or, conversely, the momentum is the integral of the force with respect to time. In particular, can you see where the slope of the graph of momentum is equal to zero? can you find the point where the slope of the graph of momentum reaches a maximum? These relationships between derivatives and integrals will form the foundation of our next subject—the study of kinematics.