Motion and Momentum

Life involves motion. This motion takes place on many size scales: from the motion of an entire organism, to the motion of a single cell, to the motion of an electron in a molecule. Even stability or equilibrium ultimately involves the concept of motion, for it is only in response to motion that we can know if a system is in a state of stable equilibrium. The motion of molecules, cells, and organisms (but not electrons!) can all be described using the branch of physics known as Newtonian Mechanics, which was developed by Isaac Newton in his *Principia Mathematica* of 1687. In this course, we will study many applications of mechanics to biological systems.

In physics, the concept of motion is linked with the concept of momentum. In everyday speech, we say that something has “momentum” if it has a tendency to continue or progress. Here are several recent headlines from the New York Times that include the term “momentum”:

- Data Readings Converge to Show an Economy Regaining Momentum
- A Steady Path to Supreme Court as Gay Marriage Gains Momentum
- Britain’s Recovery Gains Momentum, but Wages Slip

In physics, whenever an object is in motion, that object has momentum. One of the most important laws in physics is the Law of Conservation of Momentum, which states that:

**Momentum is Conserved.**

This statement of the Law of Conservation of Momentum is, of course, completely useless, since we haven’t defined either “momentum” or “conserved.” Indeed, the definition of “momentum” has changed several times in the history of physics: first in 1905, with Einstein’s Special Theory of Relativity, and then again in the 1920’s, with the development of quantum mechanics. Even the definition of “conserved” changed in 1915 with Einstein’s General Theory of Relativity. Nevertheless, every successful physical theory since the time of Newton has had something called “momentum” that is always “conserved.” As far as we know, this conservation law is exact: no violations of the Law of Conservation of Momentum have ever been observed. Our task over the next week is to explore the meaning and utility of this fundamental law.
The Momentum of a Single Object

You may be familiar with the definition of momentum in Newtonian mechanics: if an object with mass \( m \) is moving with velocity \( \vec{v} \), then the object has a momentum \( \vec{p} \) given by the expression:

\[
\vec{p} = m\vec{v}
\]

The momentum of an object belongs to that object; each object “owns” its own momentum. Indeed, you should think of the momentum of an object as located inside that object. You will not be led astray if you conceive of momentum as something tangible that is possessed by any object that is in motion.

The above equation—the definition of momentum—is a vector equation. Since the velocity vector \( \vec{v} \) is multiplied by a scalar, \( m \), we know that the momentum \( \vec{p} \) has the same direction as the velocity, but has a magnitude equal to the magnitude of the velocity multiplied by the mass of the object. If we write out the vectors \( \vec{p} \) and \( \vec{v} \) in terms of their Cartesian components, then we can see that this concise vector equation in fact represents three individual equations:

\[
\begin{align*}
    p_x &= mv_x \\
    p_y &= mv_y \\
    p_z &= mv_z
\end{align*}
\]

As you will see, we will use these component equations more often than we will use the full vector equation. Keep in mind that the simple definition \( \vec{p} = m\vec{v} \) is in fact equivalent to these three component equations.

We haven’t yet defined precisely what we mean by mass and by velocity. For now, let’s just assume that you have some intuitive notion of mass and that you understand that the velocity of an object is the change in its position with respect to time. That is, if we represent the location of an object by its displacement vector \( \vec{r} \), its velocity is defined as:

\[
\vec{v} = \frac{d\vec{r}}{dt}
\]

Although you may consider this definition of velocity to be perfectly satisfactory, we will see that it is somewhat ambiguous, since we have not yet specified what we mean by the position of an object. How would you define the position of your body at this moment? Is it defined by the
position of your left earlobe, or your right thumb, or your belly button? Clearly, if you are running down the street, the velocity of your right thumb and the velocity of your belly button will not generally be the same. We will return to this conundrum later; for now, let’s just assume that we’re dealing with very small objects whose position (and velocity) can be considered to be well-defined. In Newtonian mechanics we usually refer to such objects as particles; these (idealized) objects are infinitesimally small and thus always have well-defined positions and velocities. We will provide better definitions of both mass and velocity later, once we have explored how these concepts are used in the definition of momentum.

From this simple definition of momentum, we can deduce the dimensions and units of momentum. The mass of an object obviously has dimensions of mass, \( M \), while the velocity is length per unit time, \( LT^{-1} \). Thus we can deduce the dimensions and SI units of momentum:

\[
\text{momentum: } \mathbf{p} = \frac{ML}{T} = \frac{kg \cdot m}{s}
\]

There is no SI unit for momentum, but a 1-kg object would have one “unit of momentum” if it were moving with a velocity of 1 meter per second. (For reference, a casual walking speed is about 1 meter per second.)

**Conservation of Momentum for a Single Object**

Now that we have a definition of momentum (at least for a single object), let’s consider what it means for momentum to be conserved. The concept of “conservation” is one of the cornerstones of contemporary science; we will discuss several important conservation laws in Physical Sciences 2 and 3. The most straightforward definition of conservation requires that we divide the universe into two parts: one part, called the *system*, which contains the object(s) of interest, and another part, called the *surroundings*, which is simply the rest of the universe. You should envision an imaginary boundary around the system that separates it from the surroundings:
In many instances, we can consider a system to be approximately isolated from the surroundings: such a system does not interact in any way with the rest of the universe. Although there are no truly isolated systems, an object in deep space (far away from gravitational interactions with any planets or stars) would be isolated to a very high degree of approximation. Given that definition of an isolated system, the Law of Conservation of Momentum is:

**The momentum of an isolated system is constant.**

We can state this conservation law in a more mathematical form as follows:

\[
\text{For an isolated system, } \frac{d\vec{p}}{dt} = 0
\]

According to this formulation, conservation of momentum means that the momentum of an isolated system does not change with time.

Now let’s examine the consequences of this Law of Conservation of Momentum for a system that contains only one object (after all, at this point we only know how to calculate the momentum of a single object). In Newtonian mechanics, we have the definition \( \vec{p} = m\vec{v} \).

Inserting that definition into the Law of Conservation of Momentum, we have:

\[
\text{For an isolated system containing only one object, } \frac{d}{dt}(m\vec{v}) = 0
\]

Since the mass of an ordinary object is constant, we can take the mass \( m \) outside of the derivative and divide both sides of this equation by \( m \). Thus, we conclude that:

\[
\text{For an isolated object, } \frac{d\vec{v}}{dt} = 0
\]

This simple equation is crucial: it says that any isolated object (that is, any object that is not interacting with anything else) will move at a constant velocity. You may recognize this as an expression of Newton’s First Law:

“An object at rest will remain at rest unless acted upon by an external and unbalanced force. An object in motion will remain in motion with the same speed and in the same direction unless acted upon by an external and unbalanced force.”

This law, often called the law of inertia, is one of the great insights of Galileo and perhaps one of the greatest intellectual leaps in human history. After all, we never observe any object to move forever in a constant state of uniform motion. Galileo argued, however, that if we could truly isolate an object from all external interactions, it would move forever in a straight line with
constant speed. Today, we understand this law as merely a consequence of the more fundamental law of the conservation of momentum.

Unfortunately, truly isolated objects are mere figments of physicists’ imaginations. We can, however, define a class of objects as being functionally isolated if they behave as if they were isolated (to some degree of approximation). The best terrestrial example of objects that are functionally isolated are objects that can slide with very little friction on a level horizontal surface. We can identify these objects by inverting the above definition:

If, for a single object, \( \frac{d\dot{y}}{dt} \approx 0 \), then the object is functionally isolated.

As examples of functionally isolated objects, consider objects sliding on ice, or floating on water, or floating on a cushion of air. In each case, the objects will move in a straight line with roughly constant speed. These objects are definitely not isolated: they participate in gravitational interactions with the Earth, and they are in contact with ice, or water, or air, but these interactions effectively cancel each other out, leaving the objects free to move with very little resistance in any horizontal direction. We will often apply the Law of Conservation of Momentum to these types of functionally isolated objects, but you should be aware that these objects are by no means truly isolated from all interactions.

**Conservation of Momentum for a System of Two Objects**

Now consider a system that consists of two objects. We assume, once again, that the system is isolated (or at least functionally isolated) from the rest of the universe:

We assume that the two objects in the system can interact with one another (otherwise, we could treat each object as belonging to its own isolated system). Here we must distinguish between two types of interactions: internal interactions, which take place between objects that are both inside the system, and external interactions, which take place between an object inside the system and an object outside the system:
Once we have defined the boundary of a system, we can immediately classify all interactions as either internal or external. In this example, we assume that the two objects do participate in internal interactions, but there are no external interactions.

We wish to apply the Law of Conservation of Momentum to this system. Our first concern is, how do we define the momentum of this system? It turns out that the total momentum of a system of two objects is simply the vector sum of the momenta of the individual objects. That is,

\[ \vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 \]

(Note that this vector equation represents three component equations: one for each Cartesian component.) If the individual objects have masses \( m_1 \) and \( m_2 \), and velocities \( \vec{v}_1 \) and \( \vec{v}_2 \), then the total momentum is:

\[ \vec{p}_{\text{tot}} = m_1 \vec{v}_1 + m_2 \vec{v}_2 \]

The Law of Conservation of Momentum can be re-stated as:

**The total momentum of the objects in an isolated system is constant.**

or:

\[ \frac{d}{dt} (\vec{p}_{\text{tot}}) = 0 \]

where we define the total momentum \( \vec{p}_{\text{tot}} \) as the vector sum of the momenta of the individual objects in the system.

Let’s consider the implications of this conservation law for a system of two particles. The total momentum is constant, so we have:

\[ \frac{d}{dt} (\vec{p}_{\text{tot}}) = \frac{d}{dt} (\vec{p}_1 + \vec{p}_2) = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \]

We can rearrange this expression to obtain:

\[ \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \]
Of course, this expression is a vector equation, so it is equivalent to three equations, one for each Cartesian component:

\[
\frac{dp_{1x}}{dt} = -\frac{dp_{2x}}{dt} \\
\frac{dp_{1y}}{dt} = -\frac{dp_{2y}}{dt} \\
\frac{dp_{1z}}{dt} = -\frac{dp_{2z}}{dt}
\]

Each of these equations is independent: the first says that the \(x\)-component of the momentum is conserved, the second that the \(y\)-component of the momentum is conserved, and the third that the \(z\)-component of the momentum is conserved.

Let’s consider a concrete example. Suppose you are standing still on the surface of the Earth. If you want to start to move forward (which we’ll take to be the positive \(x\)-direction), then you’ll need to increase the \(x\)-component of your momentum from zero to some positive number. In order to do so, you have to push the Earth in the opposite direction, so the Earth will gain an equal and opposite quantity of momentum in the negative \(x\)-direction.

As we said earlier, you can consider momentum to be a tangible quantity that is contained inside each object. Any interaction (such as your push on the surface of the Earth) can transfer momentum from one object to another. This transfer of momentum from one object to another is the hallmark of interactions between objects. Indeed, we can define interactions in these terms:

**An interaction between two objects can transfer momentum from one object to the other.**

Focusing on the transfer of momentum is extremely helpful in understanding Newtonian mechanics. Whenever you examine a system, you should always ask: where is the momentum? how is the momentum changing? what interactions are transferring the momentum? To paraphrase the famous line from *Jerry Maguire*: “Show me the momentum!”

We can now offer another statement of the Law of Conservation of Momentum:

**Momentum can be transferred from one object to another, but it can never be created or destroyed.**
From this statement, we can deduce that the momentum of any isolated (non-interacting) system must be constant. The idea of the conservation of momentum was introduced first by René Descartes in 1644; he uses the term “motion” to refer to what we now call momentum:

It is obvious that when God first created the world, He not only moved its parts in various ways, but also simultaneously caused some of the parts to push others and to transfer their motion to these others. So in now maintaining the world by the same action and with the same laws with which He created it, He conserves motion; not always contained in the same parts of matter, but transferred from some parts to others depending on the ways in which they come in contact.

This statement, which predates the publication of Newton’s *Principia Mathematica*, shows the remarkable insight of this French philosopher, who also invented the familiar Cartesian coordinate system.

**Example: Conservation of Momentum for Two Objects in One Dimension**

To understand better the meaning of conservation of momentum, let’s consider a simple example. A cannon rolls freely on wheels on a flat horizontal surface. When the cannon fires, the cannonball moves to the right while the cannon itself recoils to the left:

![Cannon Diagram](http://img.sparknotes.com/content/testprep/bookimgs/sat2/physics/0003/cannon.gif)

We can analyze this situation by using conservation of momentum. We need to define a coordinate system, so let’s say that the cannonball is moving in the positive-\(x\) direction. (Can you draw coordinate axes on the above diagram that show this coordinate system?) Since the
momentum of this functionally isolated system must be constant, the initial momentum of the whole system must be equal to the final momentum of the whole system:

\[ \mathbf{p}_{\text{initial}} = \mathbf{p}_{\text{final}} \text{ or } \mathbf{p}_i = \mathbf{p}_f \]

Let the cannon be object 1, and the cannonball be object 2. Conservation of momentum requires that:

\[ m_1 \mathbf{v}_{1,i} + m_2 \mathbf{v}_{2,i} = m_1 \mathbf{v}_{1,f} + m_2 \mathbf{v}_{2,f} \]

In this example, the initial velocities are both zero, and the final velocities have components only in the \( x \)-direction. We can focus on only the \( x \)-components of the velocities:

\[ 0 = m_1 v_{1x,f} + m_2 v_{2x,f} \]

Now we are ready to answer some questions about this system. Suppose that the mass of the cannon is \( m_1 = 1000 \text{ kg} \), and the mass of the cannonball is \( m_2 = 10 \text{ kg} \). If the cannonball moves with a final velocity of \( v_{2x,f} = 100 \text{ m/s} \) in the positive \( x \)-direction, what is the final recoil velocity of the cannon?

Since the cannon is moving only in the \( x \)-direction, we need to find only the \( x \)-component of the velocity, which is \( v_{1x,f} \). Solving for this velocity in the above expression yields:

\[ v_{1x,f} = -\left( \frac{m_2}{m_1} \right) v_{2x,f} \]

Now, substituting the values for the masses and the velocity, we find:

\[ v_{1x,f} = -\left( \frac{10 \text{ kg}}{1000 \text{ kg}} \right) (100 \text{ m/s}) = -1 \text{ m/s} \]

So the \( x \)-component of the final velocity of the cannon is \(-1 \text{ m/s} \): the cannon has a speed of 1 m/s, and is traveling in the negative \( x \)-direction (to the left). Note that the dimensions of our final answer work out as required.

There are many phenomena in which two objects start off with some initial momenta, the objects interact in some way, and the objects end up with some other final momenta. If the two objects are functionally isolated during their interaction, then their total momentum must be conserved: their initial total momentum is the same as their final total momentum. We will see some more examples of these types of problems in lecture this week.
Types of Interactions

We have defined an interaction between two objects as any physical process by which momentum can be transferred from one object to another. But what are these interactions? There are two basic categories of macroscopic interactions: interactions that require contact between the two objects, and interactions that can take place when the two objects are distant from one another.

The first major category of interactions, contact interactions, can be divided into three basic types: pushing, pulling, and friction. A pushing interaction arises because objects resist being compressed. You can push on a book because your arm resists being compressed. Conversely, a pulling interaction arises because objects resist being stretched. You can pull on a book because your arm resists being stretched. A frictional interaction arises when two objects slide against one another, or when one object moves through a fluid; friction is an interaction that tends to oppose motion. A book will not slide forever on a table because of friction between the book and the table; a boat will not sail forever through water because of friction between the boat and the water, and between the boat and the air.

The second major category of interactions are long-range interactions. These interactions can operate between objects that are not in contact. The most familiar example of a long-range interaction is gravity. Gravity is a universal attractive interaction between all objects. You, and all the objects around you, are involved in a gravitational interaction with the Earth; this interaction pulls you “down,” toward the center of the Earth. Although gravitational interactions are universal (every object interacts with every other object), these interactions are quite weak. Thus, we do not notice the gravitational interactions between, say, you and your desk. We generally only notice gravitational interactions with the Earth because the Earth is so massive. There are other types of long-range interactions, such as the interactions between electrically charged objects, or the interactions between magnets, but the gravitational interaction is the most important long-range interaction in this course.

Let’s think for a minute about interactions with the Earth. Pick up a heavy book and hold it out over the floor. Right now, the book is participating in a gravitational interaction with the Earth and a contact interaction with your hand; your hand is preventing the book from falling to
the floor. The current velocity of the book is zero, so its momentum is clearly zero. Now drop the book: it falls to the floor. As it falls, it clearly has a nonzero velocity, so its momentum is now no longer zero. So how can it fall, if momentum must be conserved? When you release the book, it is participating in only one interaction: the gravitational interaction with the Earth. Thus, according to the Law of Conservation of Momentum, momentum must be transferred between the Earth and the book! As the book falls down, the Earth must rise up a very small amount to meet it! Since the Earth is so massive, however, although its momentum is changing, the change in its velocity is far too small to measure.

**Using the Conservation of Momentum to Measure Mass**

Our understanding of the conservation of momentum now allows us to measure the mass of an object by comparing it to the mass of some other object that serves as a reference. Indeed, all measurements of mass (as of 2007) are comparative measurements: these measurements can ultimately be traced back to comparisons with the mass of a single platinum-iridium cylinder stored in a vault in France that is defined as a “standard kilogram.” A photograph of this standard kilogram is shown at right (the metal cylinder is kept inside three concentric glass bell jars).

Here’s how you could measure the unknown mass of some object by comparing it with the known mass of a reference standard. Set the two masses on a level, frictionless surface, such as the air track shown in the following photograph:
Place between the two masses a small explosive charge. (Can you see the fuse in the photograph above?) Ignite the charge, which will propel the two masses away from one another. Measure the final velocities of the two masses. Now, using the measured velocities of the two masses and the known mass of one of the objects, can you figure out how to calculate the unknown mass of the other object?

We have come full circle, back to our initial question of how to define mass. By using the Law of Conservation of Momentum, we can offer a technique for measuring the mass of an object. We have not been able to tell you what mass is (indeed, that question remains unanswered to this day!), but we can tell you how to measure mass and how to use mass. If you like, you may consider the mass of an object to be simply “the thing that you multiply by its velocity in order to find its momentum.”

**Systems of Many Objects; The Center of Mass**

If a system contains many particles, the total momentum of the system is given by the vector sum of the momenta of all the particles in the system:

\[ \vec{p}_{\text{total}} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots \]

The momentum of each individual particle is given by our basic definition, \( \vec{p} = m \vec{v} \), so we can expand the right-hand side of that equation:

\[ \vec{p}_{\text{total}} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots \]

But what about the left-hand side of that equation? Can we say that the total momentum of the system is given by something like \( \vec{p}_{\text{total}} = M_{\text{total}} \vec{v}_{\text{system}} ? \) Let’s just define the total momentum in that fashion and see where that definition leads us. We have:

\[ \vec{p}_{\text{total}} = M_{\text{total}} \vec{v}_{\text{system}} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots \]

Now we must find some reasonable and consistent definitions of \( M_{\text{total}} \) and \( \vec{v}_{\text{system}} \). It seems that \( M_{\text{total}} \) should be the total mass of the system, that is:

\[ M_{\text{total}} = m_1 + m_2 + m_3 + \ldots \]

We next must define \( \vec{v}_{\text{system}} \). What is the velocity of a system that consists of many particles, each of which is moving at its own velocity? According to the equations above, we have:
Let’s think about this equation for a bit. It seems that the velocity of a system is not defined as a simple average of the individual velocities, but is rather a “mass-weighted average” of the velocities of the individual particles that make up the whole system. For instance, if the mass of one particle is much greater than the mass of all the others, then the velocity of the entire system will be approximately equal to the velocity of that single, massive particle.

We now have a reasonable definition of the velocity of a system; we must now find an intuitive understanding of this velocity. The total velocity of a system is the velocity of what, exactly? We know that velocity is generally defined as the rate of change of position over time:

\[ \vec{v} = \frac{d\vec{r}}{dt} \]

Perhaps the “velocity of the system” is the velocity of some point that represents the motion of the entire system; we will call this point (for now!) simply \( \vec{r}_{\text{system}} \), the “position of the whole system”:

\[ \vec{v}_{\text{system}} = \frac{d\vec{r}_{\text{system}}}{dt} \]

Let’s insert these definitions of velocity into our expression for the total velocity:

\[ \frac{d\vec{r}_{\text{system}}}{dt} = \left( \frac{1}{M_{\text{total}}} \right) \left( m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + m_3 \frac{d\vec{r}_3}{dt} + \ldots \right) \]

From this expression, it appears that we should define \( \vec{r}_{\text{system}} \), the “position of the system,” as:

\[ \vec{r}_{\text{system}} = \left( \frac{1}{M_{\text{total}}} \right) (m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots) \]

As we found in class, if you take the time rate of change of both sides of this equation by taking the time derivative \( \frac{d}{dt} \) of both sides, you get the expression for velocity just above.

We can now revert to more conventional notation; we confess that the “position of the system” \( \vec{r}_{\text{system}} \) is usually called the center of mass of the system, and is usually notated \( \vec{r}_{\text{CM}} \). The center of mass of any system is thus defined by:

\[ \vec{r}_{\text{CM}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots}{M_{\text{total}}} \]
We likewise define the velocity of the center of mass (what we were previously calling the “velocity of the system”) as:

\[ \vec{v}_{\text{CM}} = \frac{d\vec{r}_{\text{CM}}}{dt} \]

The motivation for these definitions is, as you have seen, that the total momentum of any system is equal to the total mass of that system times the velocity of the center of mass of the system. Going back to where we started, we find that:

\[ \vec{p}_{\text{total}} = M_{\text{total}} \vec{v}_{\text{CM}} \]

In other words, the momentum of any complex system can be reduced to the momentum of the center of mass of that system. This fact allows us to treat complicated objects as if they were simple objects that move with a single velocity. The center of mass of a standing human is roughly near the belly button, so when we consider the velocity of a runner, we do indeed mean (more or less) the velocity of her belly button, not the velocity of her right thumb or left earlobe. We now have an unambiguous definition of the velocity of any object (no matter how complex):

**The velocity of any object or system is defined as the velocity of its center of mass.**

Finally, we can apply conservation of momentum to obtain:

**For any isolated object or system, \( \frac{d\vec{v}_{\text{CM}}}{dt} = 0 \)**

This conclusion is remarkable: no matter how complex the object or system, as long as it is isolated, its center of mass will move in a straight line at a constant speed!