SUMMARY OF DEFINITIONS, AXIOMS, AND PROOFS FOR THE WEEK OF 15 FEBRUARY 2016

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Abstract. Given the weather conditions this week, coupled with the nature of this week’s learning objectives, I’ve prepared this document that you should use in completing homework as well as a study guide as we move forward.

1. Definitions

We use the following definitions as starting points for our exploration of mathematical proof. Our domain of interest includes facts about the integers, although this is an informal introduction to a more detailed study of the integers that we will undertake in the near future.

- Note, in the following we do not formally define “integer;” this will be done later. Assume that by the term “integer” we mean the whole numbers, including the negative whole numbers and zero.
- We define only those properties of the Rational numbers required for the present discussion.

Definition (Even). The integer $n$ is even if and only if it can be written as $n = 2k$, for some integer $k$.

Definition (Odd). The integer $n$ is odd if and only if it can be written as $n = 2k + 1$, for some integer $k$.

Definition (Rational numbers). The rational numbers, written $\mathbb{Q}$, is the infinite set of numbers written $a/b$ where $a$ and $b$ are integers, $b \neq 0$. Written formally:

$$\{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$

In addition to the definitions above, we rely upon the following axioms, which are postulates that are accepted as true without proof.

Axiom (Closure of addition/subtraction). Addition (and its inverse, subtraction) is closed over the integers: Let $a$ and $b$ be any integers, then $a + b$ or $a - b$ is an integer.

Axiom (Closure of multiplication over the Integers). Multiplication is closed over the integers. Given integers $a$ and $b$, then $ab$ is an integer.

Axiom (Associativity of addition and multiplication). Addition (and subtraction) and multiplication over the integers obey the associative laws.
Axiom (Commutativity of addition and multiplication). Addition and multiplication are commutative over the integers.

Axiom (Distributivity of operations). Addition (subtraction) and multiplication distribute over the integers.

Axiom (Most reduced form for rationals). The rational number \( a/b \) may be written in its “most reduced form,” which means that it can be expressed such that 1 the greatest common divisor between \( a \) and \( b \).

2. Common propositions and their proofs

We explored some common propositions this week. I have included some of these that you may use as a reference for your homework as well as a review of what was done in class.

Proposition. The sum of two even integers is even.

Proof. We prove this directly. Assume that both \( a \) and \( b \) are even, then \( a = 2k \) and \( b = 2l \) for integers \( k \) and \( l \). Hence their sum is \( a + b = 2k + 2l = 2(k + l) \). And this last term is an even integer which was to be shown.

Proposition. The sum of an even integer and an odd integer is an odd integer.

Proof. We prove this directly. Let \( a \) be an even integer and \( b \) an odd integer, then \( a = 2m \) and \( b = 2n + 1 \), for integers \( m \) and \( n \). Thus \( a + b = 2m + 2n + 1 = 2(m + n) + 1 \), and this last term is an odd integer, which needed to be shown.

Proposition. No integer is both even and odd.

Proof. BWOC assume that the proposition is false; then there is at least one integer that is both even and odd. Let \( n \) be such an integer. Then, \( n = 2k \) and \( n = 2l + 1 \), for integers \( k \) and \( l \). This assumption gives \( 2k = 2l + 1 \) and \( 2k = 2l + 1 \), for integers \( k \) and \( l \). This assumption gives \( 2k = 2l + 1 = 2(k - l) = 1 \). But no integer multiplied by 2 can produce 1, which is a contradiction.

Because the assumption that the proposition was false led to a contradiction, we assert that the proposition is true.

The following proposition is used often—especially in showing the rationality or irrationality of certain expressions.

Proposition. For any integer \( n \): if \( n^2 \) is even, then \( n \) is even.

Proof. We use the contrapositive, which means that we re-write the implication as “if \( n \) is odd, then \( n^2 \) is odd.” Let \( n = 2a + 1 \) for some integer \( a \). Then \( n^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1 \), and this last term is an odd integer, which needed to be shown.

Here’s a proof of a biconditional, which requires that we show that the implication and its converse are both true.
Proposition. Show that for any positive integer \( n \) that \( n \) is even if and only if \( 5n + 6 \) is even.

Proof. We need to show two things. The first is for any positive integer \( n \), if \( n \) is even then \( 5n + 6 \) is even. Assume \( n \) is even, then \( n = 2k \) for some integer \( k > 0 \). Thus, \( 5n + 6 = 5(2k) + 6 = 10k + 6 = 2(5k + 3) \), and this last term is an even integer, which was to be shown.

We now show the converse. We do this by showing that the inverse of the original proposition is true: if \( n \) is odd then \( 5n + 6 \) is odd. Since \( n \) is odd, we have \( n = 2k + 1 \), for some integer \( k \geq 0 \). This gives \( 5n + 6 = 5(2k + 1) + 6 = 10k + 10 + 1 = 2(5k + 5) + 1 \), and this last term is an odd integer, which needed to be shown.

Having shown that both the implication and its converse are true, the proposition is true.

We postpone any in-depth treatment of the rationals (and the irrationals and reals) for a few weeks. We can, however, construct the following proof, using only the definitions and axioms that are given in this paper.

Proposition. The equation \( r^3 + r + 1 = 0 \) has no rational roots.

Proof. BWOC assume otherwise, then let \( a/b \) be a root where \( a, b \) are integers, \( b \neq 0 \), and \( a \) and \( b \) can be expressed in their most reduced form, meaning that they can be re-written to have a greatest common divisor of 1. This assumption gives:

\[
\left( \frac{a}{b} \right)^3 + \frac{a}{b} + 1 = 0.
\]

Removing the denominator gives:

\[
a^3 + ab^2 + b^3 = 0.
\]

Because \( a \) and \( b \) have a common denominator of 1 they cannot both be even. But in the event that either is odd and the other even, the sum above is odd. This would mean that an odd number equals 0, which is an even number, and this is a contradiction.

Because the assumption above led to a contradiction, we conclude that the proposition is true.