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Constructing Force-Closure Grasps

Abstract

This paper presents fast and simple algorithms for directly constructing force-closure grasps based on the shape of the grasped object. The synthesis of force-closure grasps finds independent regions of contact for the fingertips, such that the motion of the grasped object is totally constrained. A force-closure grasp implies equilibrium grasps exist. In the reverse direction, we show that most nonmarginal equilibrium grasps are force-closure grasps.

1. Introduction

1.1. How Should the Fingers be Placed?

Grasping an object is exerting force and moment on the object to move it or to keep it in stable equilibrium. Grasping is also constraining the motion of the object by a set of contacts. These two descriptions are dual to each other. They correspond respectively to a force-closure grasp and to zero total freedom of the object. A grasp on an object is force-closure if and only if we can exert, through the set of contacts, arbitrary force and moment on this object. Equivalently, any motion of the object is resisted by a contact force.

The fingertip contacts are modeled as hard or soft point contacts, with or without friction. The forward problem is to analyze whether a grasp, defined by a set

of contacts, is force-closure or not. The reverse problem is to find places to put the fingertips such that the grasp is force-closure. The synthesis method we develop finds large independent regions of contact for the fingertips. Figure 1 shows examples of force-closure grasps in two (2D) and three dimensions (3D). The independent regions of contact for the fingertips are highlighted by bold segments and circles. The focus will be 2D and 3D grasps, respectively, on polygonal and polyhedral objects.
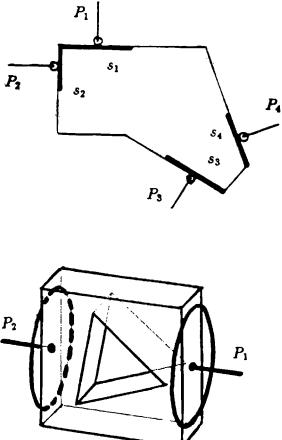
1.2. Main Results

The main results of this paper are as follows:

1. A grasp is described as the combination of individual contacts, which in turn are modeled as the combination of a few primitive contacts. They are hard or soft point contacts with or without friction. A contact over a finite segment or surface has a very compact representation if its normal is constant. This explains why the synthesis of force-closure grasps is very simple for polygonal and polyhedral objects.
2. The algorithms for constructing force-closure grasps on polygons and polyhedra are direct, fast, and simple. We find not only single grasps but the complete set of all force-closure grasps on a set of edges and faces. We can also construct the independent regions of contact for the fingers. The construction is exponential in the minimum number of required fingers and polynomial in the number of total fingers.
3. We show that nonmarginal equilibrium grasps are force-closure grasps. This proof supports a very simple heuristic for grasping objects with two fingers: "Increase friction and compliance at the contact by covering the fingertips with soft rubber. Then grasp the object on two opposite sides."

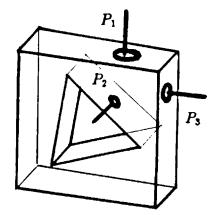*Fig. 1. Examples of force-
closure grasps in 2D and 3D.*

1.3. Other Related Works

Related works can be grouped as follows:

1. Force-closure grasps: Force-closure and total freedom capture the main constraint between the fingers and the grasped object. Ohwovoriole analyzed the geometry of the different repelling screw systems and used the results to analyze systems of contracting bodies, such as an object grasped by a set of fingers, or a pin being inserted into a hole (Ohwovoriole 1980, 1984). Related to force-closure is the solution of systems of linear inequalities (Goldman and Tucker 1956).
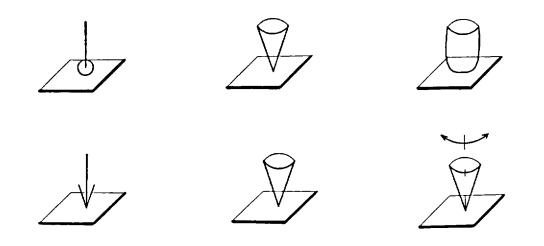
2. Form-closure grasps: A grasp is form-closure if the grasped object is totally constrained by the set of contacts, irrespective of the magnitude of the contact forces. Reuleaux (1875) proved that a 2D grasp needs at least four point contacts for form-closure. Lakshminarayana (1978) showed that a 3D grasp needs at least seven point contacts. Form-closure can be viewed as force-closure with frictionless contacts only. Mishra, Schwartz, and Sharir (1986) showed that no form-closure grasp exists on finite surfaces of revolution, on infinite planes, cylinders, or helical surfaces.

Fig. 2. Primitive contacts in
2D and 3D.

2. Contacts and Grasps

2.1. Primitive Contacts

The contacts between the fingertips and the object can be modeled as frictionless point contacts, hard-finger contacts, or soft-finger contacts. The three primitive contacts and their wrench convexes are respectively shown on the top and bottom rows of Fig. 2. From left to right the figure shows:
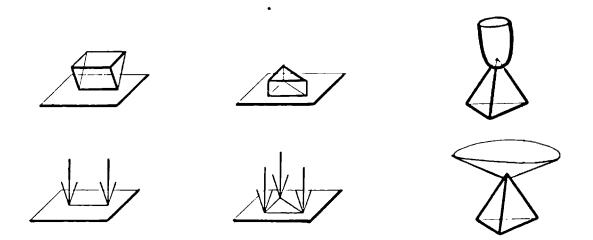
1. Frictionless point contact: The finger can only exert a normal force through the point of contact. The wrench convex has a single wrench, with line of action going through the point of contact and with direction the negative of the contact normal.
2. Hard-finger contact: This is a point contact with friction. The finger can exert any force pointing into the friction cone at the point of contact. This 2D (resp. 3D) friction cone describes the wrench convex, which mathematically is the convex sum of two (resp. infinitely many) generating wrenches. The 3D friction cone is usually approximated by a polyhedral convex, with vertex at the point of contact (Kerr 1985).
3. Soft-finger contact: The friction over the area of contact allows the finger to exert pure torques in addition to pure forces pointing into

the friction cone. A 3D soft finger can exert torques in both directions, about the normal axis at the point of contact. So the wrench convex is described by a one-sided friction cone plus a two-sided torque. This two-sided torque has no effect in a 2D grasp, so the soft finger is reduced to a convex sum of hard fingers over the small segment of contact.

Any complex contact can be described as a convex sum of the above primitive contacts. Figure 3 shows an edge contact without friction whose wrench convex is the convex sum of two wrench convexes; each describes the frictionless point contact at one end of the edge of contact. Similarly, a face contact is the convex sum of point contacts at the vertices of the face. The wrench convex of the polygonal face is minimally described as the convex sum of point contacts that are on the convex hull of the face. The last column describes a soft contact touching a vertex and its wrench convex computed from the convex sum of all friction cones over the small contact patch between the soft finger and the vertex. This wrench convex can be approximated by a friction cone with a much larger sector.

2.2. Dual Representations for Grasp

A wrench convex describes the range of force and moment that can be applied on the object. A twist convex reciprocal or repelling to a wrench convex de-

Fig. 3. Complex contacts.

scribes the range of motions for which the object can move freely or break contact. We can sum wrench convexes from all the contacts, or intersect the corresponding twist convexes, to find the resulting wrench or twist convex of the grasp. The goal here is to get a wrench convex that spans the whole space of force and moment, or a null twist convex.

Wrench and twist convexes are two dual representations for contact and grasp. For planning grasps, wrench convexes are definitely a more efficient representation, since generating wrenches can be deduced readily from the type and point of contact as above, and we can just take the union of all the generating wrenches to describe the grasp. From now on, a grasp G will be specified by a set of points of contact $P_1, \ldots, P_n$, or a set of wrenches W generating the wrench convex.

The force-closure problem can be formulated as the solution of a system of linear inequalities $W^T\hat{t}^s \geq 0$, where each column of the matrix W is a contact wrench. The product of the ith row of W^T and the spatial transpose of the twist $\hat{t}^s$ is the spatial dot product $\hat{w}_i \hat{t}$, describing the virtual work of the wrench $\hat{w}_i$ against the twist $\hat{t}$. A twist is called reciprocal, repelling, or contrary to a wrench if and only if the spatial dot product of the twist and the wrench is, respectively, zero, positive, or negative (Ohwovoriole 1980).

A grasp between an object and many fingers is different from a linked chain or platform, such as a robot arm, in that forces can be applied in one direction only. The fingers can only push, not pull, on the object because there is no glue between the object and the fingers. In a revolute (resp. prismatic) arm, torques (resp. forces) can be applied in both directions at the joints of the arm. This is the reason why the kinematic constraints in a grasp must be described in terms of convexes – positive combinations of contact forces — instead of subspaces – linear combinations of spatial vectors — as in the analysis of arm kinematics (Featherstone 1983).

A grasp is force-closure if and only if the system of linear inequalities $W^T\hat{t}^s \geq 0$ has no solution. In general, vector closure in an n-dimensional space needs at least $n + 1$ vectors.

THEOREM 1 *(GOLDMAN AND TUCKER)* *In an n-dimensional vector space, a set of vectors V is vector-closure if and only if V has at least $n + 1$ vectors $(v_1, \ldots, v_{n+1})$ such that*

1. *n of the $n + 1$ vectors are linearly independent.*
2. *A strictly positive combination of the $n + 1$ vectors is the zero vector.*

$$\sum_{i=1}^{n+1} \alpha_i v_i = 0, \qquad \alpha_i > 0. \tag{1}$$

The first statement expresses the necessary and sufficient condition for no homogeneous solution to the system $V^T x \geq 0$. V is the matrix with vectors v_i as columns. The number of independent vectors must be equal to the dimension of the vector space. The sec-

Fig. 4. A geometrical view of force-direction closure in 2D.

ond statement expresses the additional necessary and sufficient condition for no particular solution. The theorem is just a slightly different form of a lemma (Lemma 6) proved by Goldman and Tucker (1956).

Polygonal (resp. polyhedral) objects have edges (resp. faces) with constant normal, so the force-closure problem can be split into two independent subproblems: (1) force-direction closure, which tests that the friction cones from the finger contacts span all directions, and (2) torque-closure, which makes sure that coupling of the contact forces creates all pure torques. The placement of a finger within a straight edge (resp. planar face) affects only the torque-closure condition. For hard-finger (resp. soft-finger) contacts with friction, the torque-closure condition in 2D (resp. 3D) has a very simple geometric formulation, and this results in fast and simple algorithms for constructing the independent contact regions. We describe in detail the analysis and synthesis of force-closure grasps in 2D, and give extensions in 3D.
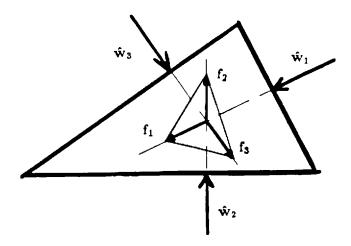
3. Resisting Translation and Rotation

3.1. Force-Direction Closure

THEOREM 2 *A set of planar wrenches W can generate force in any direction if and only if there exists a three-tuple of wrenches $(\hat{w}_1, \hat{w}_2, \hat{w}_3)$ whose respective force directions f_1, f_2, f_3 satisfy*

  1. Two of the three directions f_1, f_2, f_3 are independent.
  2. A strictly positive combination of the three directions is zero: $\alpha f_1 + \beta f_2 + \gamma f_3 = 0$.

The first (resp. second) condition corresponds to no homogeneous (resp. particular) solution to the system $W^T \hat{t}^s \geq 0$, where twist $\hat{t} = (0, d_x, d_y)^T$ is an infinitesimal translation of the object. Theorem 2 can be captured in a more suggestive and compact way (Fig. 4) as follows.

COROLLARY 1 *A set of planar wrenches W can generate forces in any arbitrary direction if and only if*
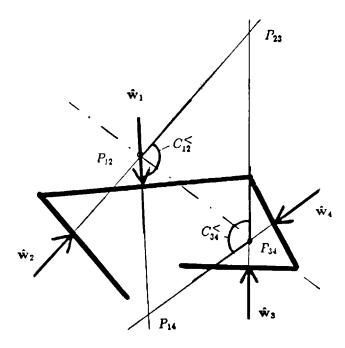

there exists a three-tuple of force-direction vectors (f_1, f_2, f_3) whose endpoints draw a nonzero triangle that includes their common starting point.

Force-direction closure in 3D needs four directions, three independent and the fourth a negative combination of the first three. An example is the four normal directions of a tetrahedron. Most grasps have friction at the fingertips, and force-direction closure is usually achieved by having two friction cones whose sectors counterintersect.

3.2. Torque-Closure

Torque-closure in 2D can be achieved by creating enough friction on some axis of rotation of the object. The friction between the rotating object and its supporting axis will create a torque that resists any clockwise or counterclockwise rotation of the object. Unfortunately, in most grasp configurations, we have only point contacts, and through a point contact a finger can exert only pure force, and not torque on the object. A more interesting problem is to achieve torque-closure with only pure forces.

THEOREM 3 *A set of planar forces W can generate clockwise and counterclockwise torques if and only if there exists a four-tuple of forces $(\hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4)$ such that the following hold.*

Fig. 5. A geometrical view of
torque-closure in 2D.

1. *Three of the four forces have lines of action that do not intersect at a common point or at infinity.*
2. *Let $f_1, \ldots, f_4$ be the force directions of $\hat{w}_1, \ldots, \hat{w}_4$. Let p_{12} (resp. p_{34}) be the point where the lines of action of $\hat{w}_1$ and $\hat{w}_2$ (resp. $\hat{w}_3$ and $\hat{w}_4$) intersect. There exist $\alpha, \beta, \gamma, \delta$, all greater than zero, such that*

$$p_{34} - p_{12} = \pm(\alpha f_1 + \beta f_2) = \mp(\gamma f_3 + \delta f_4).$$

The first (resp. second) condition corresponds to no homogenous (resp. particular) solution to the system $W^T \hat{t}^s \geq 0$, where twist $\hat{t} = (\delta_z, d_x, d_y)^T$ is an infinitesimal rotation of the object. For a detailed proof see Nguyen (1986).

Theorem 3 can be formulated in more geometrical terms as follows.

COROLLARY 2 *A set of planar forces W can generate clockwise and counterclockwise torques if and only if there exists a four-tuple of forces $(\hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4)$ such that the segment $P_{12}P_{34}$, or $P_{34}P_{12}$, points out of and into the two cones $C_{12}^{<}$, $C_{34}^{<}$ formed by the two pairs $(\hat{w}_1, \hat{w}_2)$ and $(\hat{w}_3, \hat{w}_4)$.*

From Fig. 5, one can check for torque-closure in the plane by drawing a quadrilateral delimiting the over-lapping region of the two cones $C_{12}^{<}$, $C_{34}^{<}$. From this quadrilateral, one can generate clockwise and counterclockwise torques from positive combinations of the four pure forces. Unfortunately, there is no simple geometric interpretation for torque-closure in 3D because nonparallel lines in space generally do not intersect. Torque-closure in 3D needs at least seven forces.

The necessary and sufficient condition for force-closure is contained in Theorems 2 and 3. If we assume that through any contact we can only exert force and not torque, then Theorem 3 subsumes Theorem 2. A translation can be viewed as a rotation with point of rotation at infinity. So, if there is no free rotation for the grasped object constrained by a set of contact forces, then there exists no free translation. Thus Corollary 2 describes the geometrical necessary and sufficient condition for force-closure with planar forces only.
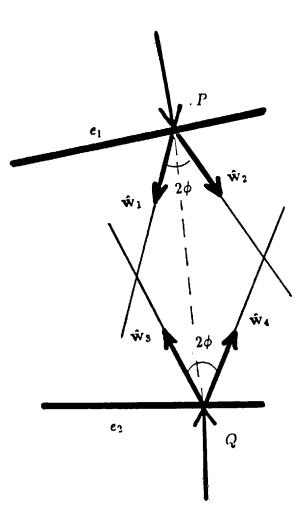
4. Finding Force-Closure Grasps

4.1. Two Opposing Fingers

COROLLARY 3 *Two point contacts with friction at P and Q form a planar force-closure grasp if and only if the segment PQ, or QP, points strictly out of and into the two friction cones respectively at P and Q or, formally, $\arg(q - p) \in \pm(\mathscr{C}_1 \cap -\mathscr{C}_2)$ where $\mathscr{C}_1$ and $\mathscr{C}_2$ are angular sectors of friction cones, respectively, at e_1 and e_2.*

Proof: This is a well-known fact of planar mechanics. However, we prove Corollary 3 by using a reduction from a grasp with two point contacts with friction to a grasp with four point contacts without friction. A friction cone at P (resp. Q) is equivalent to two forces $\hat{w}_1$, $\hat{w}_2$ (resp. $\hat{w}_3$, $\hat{w}_4$) along the edge of the friction cone and going through P (resp. Q); see Fig. 6. We recognize that point P (resp. Q) is nothing more than point P_{12} (resp. P_{34}). So Corollary 3 is a special case of Corollary 2.

LEMMA 1 *The set of all force-closure grasps with fraction on two edges e_1, e_2 is completely described by edges e_1, e_2 and the counteroverlapping sector $\pm(\mathscr{C}_1 \cap -\mathscr{C}_2)$.*

Fig. 6. Finding grasps with
friction.

grasp is force-closure. This partially explains why people grasp objects at edges and corners, and why the contacting surface of human fingers need to be soft rather than hard like the fingernails.
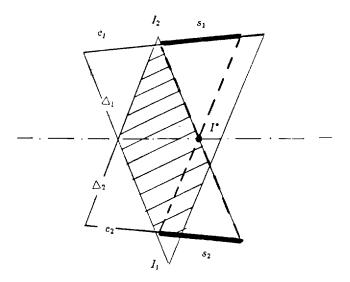
4.2. Nonmarginal Equilibrium

A force-closure grasp implies that equilibrium grasps exist because zero force and moment is spanned by the set of contact forces. In the reverse direction, it turns out that most nonmarginal equilibrium grasps are force-closure grasps. We first prove the simple case of two point contacts with friction in 2D.

COROLLARY 4   *Let G be a 2D grasp with at least two distinct point contacts with friction. G is a force-closure grasp if it is an equilibrium grasp and has contact forces pointing strictly within their friction cones.*

Proof: The two friction cones give three lines of force that are not all parallel, because the friction cones are not null. These three lines of force do not all intersect at the same point because the two points of contact are distinct. So, we have three planar wrenches with independent spatial vectors. The set of contact wrenches is also force-closure, or vector-closure, if there exists a strictly positive combination of four contact wrenches that results in the zero wrench, or equilibrium. The coefficients of the four contact wrenches are strictly positive because the contact forces point strictly inside their respective friction cones.

A 3D grasp with two opposite hard fingers cannot resist rotations about the segment joining the two points of contact. To be force-closure, a nonmarginal equilibrium grasp needs at least two distinct soft-finger contacts or three distinct hard-finger contacts with friction. We can extend this result to more complex contacts, such as edge and face contacts with friction. For example, an edge contact with friction is equivalent to two hard-finger contacts, so force-closure needs at least two edge contacts or one edge contact and one hard-finger contact.

In a 3D grasp, by coupling opposite contact forces, we can generate arbitrary pure torques perpendicular to the segment PQ joining the two points of contact. These torques can be generated if and only if the segment PQ, or QP, points strictly out of and into the two friction cones, respectively, at P and Q. If the contacts at P and Q are soft-finger contacts, then the pure torques about P and Q have projections on the segment PQ, so the grasp is also torque-closure. Corollary 3 also applies to a 3D grasp with two soft-finger contacts.

We have seen that a soft finger contacting an acute vertex can be approximated as a point contact with a much wider friction cone. From Corollary 3, the larger the friction cones at the points of contact, the greater is the likelihood that they counteroverlap or that the

*Fig. 7. Finding the indepen-
dent regions of contact.*

5. Finding Independent Regions of Contact

In task planning, we are interested in finding grasps
that require as little accuracy as possible. One aspect
of that goal is to have grasps such that the fingers can
be positioned independently from each other, not at
discrete points, but within large regions of the edges or
faces.

5.1. 2D Grasp with Two Point Contacts with Friction

Corollary 3 allows us to cast the problem of finding
the independent regions of contact on two edges into a
problem of fitting a two-sided cone cutting these two
edges into two segments of largest minimum length
(Fig. 7). A two-sided cone with vertex at I and opposite
sectors $\pm\mathscr{C}$, denoted by $C^\times(I, \pm\mathscr{C})$, is nothing but the
union of the opposite cones $C^<(I, \mathscr{C})$ and $C^<(I, -\mathscr{C})$.

ALGORITHM 1   *The independent regions of contact on
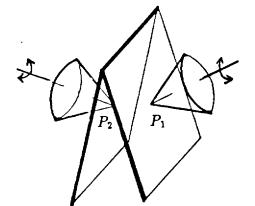two edges e_1 and e_2 can be constructed as follows:*

1. *Find the counteroverlapping sector $\pm\mathscr{C} =
   \pm(\mathscr{C}_1 \cap -\mathscr{C}_2)$ from the sectors $\mathscr{C}_1$ and $\mathscr{C}_2$ of the
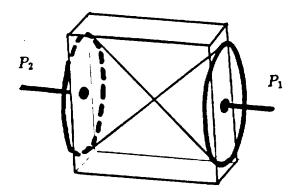   friction cones on edge e_1 and e_2.*

2. *Find the two-sided cone $C^\times(I_1, \pm\mathscr{C})$ that cuts
   all of edge e_1 and very little or none of edge e_2.
   We get a triangle $\triangle_1$ formed by edge e_1 and
   vertex I_1. This triangle represents the set of ver-
   tices I, where the two-sided cone $C^\times(I, \pm\mathscr{C})$
   monotonically cuts the larger segment e'_1 and
   the smaller segment e'_2 as we move from edge e_1
   to edge e_2. Similarly, we find the two-sided
   cone $C^\times(I_2, \pm\mathscr{C})$ such that this later cuts ex-
   actly the edge e_2 and very little or none of edge
   e_1. We get a triangle $\triangle_2$ formed by edge e_2 and
   vertex I_2.*

3. *Find the tradeoff region for vertex I by inter-
   secting triangles $\triangle_1$ and $\triangle_2$. The tradeoff region
   describes the locus of vertex I for which the
   two-sided cone $C^\times(I, \pm\mathscr{C})$ cuts both edges e_1
   and e_2 into segments e'_1 and e'_2. The length of
   the independent segments e'_1 and e'_2 is propor-
   tional to the distance from vertex I of the two-
   sided cone to the respective edges.*

4. *We cut the tradeoff region with the bisector of
   edges e_1 and e_2. The intersection is the locus of
   vertex I for which segments e'_1 and e'_2 have the
   same length. The optimal vertex $I*$ is any-
   where on the intersecting segment, or at one of
   the two endpoints of this segment, depending
   on whether or not the two edges are parallel. If
   no intersecting segment exists, then the optimal
   vertex is the point of the tradeoff region that is
   nearest to the bisector.*

5. *From the optimal vertex $I*$, the independent
   regions of contact s_1 and s_2 are found by inter-
   secting the two-sided cone $C^\times(I*, \pm\mathscr{C})$ and the
   grasping edges e_1 and e_2.*

The computation of the optimum independent
regions of contact for two point contacts with friction
on two edges takes about 0.05 s. The code is written in
Zeta Lisp and compiled and run on a Symbolics ma-
chine.

5.2. 3D Grasps with Two Soft-Finger Contacts

The polyhedral faces have constant normals, so the
force-direction closure condition reduces to a simple

Fig. 8. Force-closure with
two soft-finger contacts.

Fig. 8. Force-closure with two soft-finger contacts.

test of the angle between the two plane normals. Once the force-direction closure is satisfied, the two friction cones counteroverlap, and the counteroverlapping sector is $\pm(\mathscr{C}_2 \cap \mathscr{C}_1)$. With soft-finger contacts, the torque-closure condition is satisfied if and only if the segment $P_1 P_2$ has orientation inside the counteroverlapping sector. The independent contact regions can be constructed by intersecting the two faces with a two-sided cone, having cone angle $\pm(\mathscr{C}_2 \cap \mathscr{C}_1)$ and a vertex between the two faces (Fig. 8).
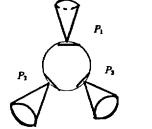
The construction is similar to its 2D analogue given in Algorithm 1. The faces are approximated by their bounding circular disks. The sector $\mathscr{C}_2 \cap -\mathscr{C}_1$ is approximated by the maximum cone inside $\mathscr{C}_2 \cap -\mathscr{C}_1$. The two-sided cone is positioned between the two disks bounding the two faces. The intersections between the two disks and their respective cones give the independent contact regions. The approximate computation of the independent regions for two soft-finger contacts on two faces takes about 0.05 s.
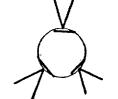
If the face has holes or if it is nonconvex, then the circular disk bounding the face no longer preserves the compactness or convexity property of the face. A nonconvex face is approximated by a set of overlapping circular disks; each disk approximates a local convex region of the face. Local convex regions can be computed from the Voronoi diagram of the face (Shamos 1978). They can be approximated by the generalized cones between opposing edges as in Nguyen (1984).

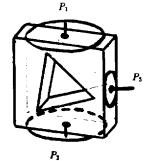## 5.3. 3D Grasps with Three Hard-Finger Contacts

The force-closure condition becomes a constraint on the relative configuration of the friction cones. The independent contact regions are constructed by cutting the two-sided cones and the faces of contact. The force-closure grasps with three hard-finger contacts can be split into four classes (Fig. 9), depending on the number of friction cones that pairwise counteroverlap.
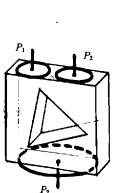
1. The first grasp has no pair of counteroverlapping cones. An example is a three-point grasp on a ball with very little friction. The three grasp points are symmetrically placed on a circle that has the same center as the ball. Note that the ball will slip away from the fingers if one of the three contact points is removed.
2. The second grasp has one pair of counteroverlapping cones, from the top and bottom contacts. The third contact contributes a torque component about $P_1 P_2$. This contact can be removed without having the object slip from the fingers.
3. The third grasp has two pairs of counteroverlapping cones. The second contact serves as a pivot when either the first or the third contact is added or removed. An example is a three-point grasp on two parallel faces, with two of the fingers on the same face.
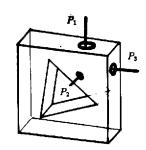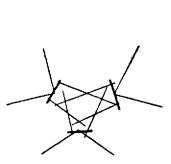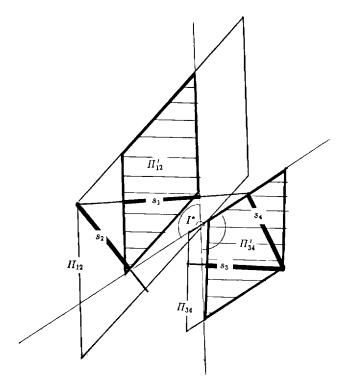
4. The fourth grasp has three pairs of counter-overlapping cones. All three contacts can be used as pivots, and any fingertip can be removed or added while the other two grasp the object. This grasp exists only if the coefficient of friction is greater than tan 30°.

The foregoing classification arises directly from the geometric construction of the independent contact regions. The classification highlights the similarity and difference between grasps with soft fingers and grasps with hard fingers. We can change from one grasp to another by searching for a sequence of two-point and three-point grasps. The two-point grasps are force-closure if the fingertips are soft. We can also synthesize virtual springs at the fingertips such that the two-point and three-point grasps are stable (Nguyen 1987). In other words, one grasp is changed to another by a sequence of stable force-closure grasps. Only one finger is removed or added at a time, while the other two fingers maintain a stable force-closure grasp on the object.

5.4. 2D Grasps with Four Frictionless Contacts

With Corollary 2, the problem of finding the independent regions of contact on three or four edges becomes a problem of fitting a two-sided cone between two parallelograms. Figure 10 illustrates the fitting of a two-sided cone between the two parallelograms Π_{12} and Π_{34}. The two-sided cone has vertex at I and sector $\pm\mathscr{C} = \pm(\mathscr{C}_{12} \cap -\mathscr{C}_{34})$. This two-sided cone cuts the two parallelograms Π_{12} and Π_{34} into two disjoint regions, for which Corollary 2 is satisfied for any pair of points (P_{12}, P_{34}) from these two regions. Even better, we restrict the two disjoint regions to two smaller parallelograms Π'_{12} and Π'_{34} so that the point of contact P_1 (resp. P_3) can be independently placed from P_2 (resp. P_4). The problem is to find the optimum position of the vertex I such that Π'_{12} and Π'_{34} have largest minimum distance between parallel edges. The independent contact regions are found by backprojecting the smaller parallelogram Π'_{12} on edges e_1 and e_2, and similarly for Π'_{34}.

Fig. 10. Independent regions for four frictionless point contacts in 2D.

As we translate one of the edges of the cone $C^\times(I, \pm\mathscr{C})$, then Π'_{12} and Π'_{34} vary monotonically in opposite directions. This monotonicity allows us to consider only a finite number of boundary cases. We partition the plane into regions R_i's depending on how the two-sided cone cuts Π_{12}. In each region R_i, the loci of vertex I, for which Π'_{12} has constant area, form parallel lines. We find similar regions R_j's and loci for Π_{34}. The problem then reduces to computing all pairwise intersections $R_i \cap R_j$ from the two sets of regions and, for each intersection, computing the locally optimum vertex I from the two corresponding loci. The computation of the four optimal independent regions of contact takes about 0.25 s on a Symbolics machine.

The synthesis of the four independent segments of contact can be viewed as finding four convexes $C_1^\lessgtr, \ldots, C_4^\lessgtr$, such that any four-tuple of wrenches $(\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_4)$ is vector-closure or

$$\sum_{i=1}^{4} \alpha_i \hat{\mathbf{w}}_i = \hat{\mathbf{0}}, \qquad \alpha_i > 0, \tag{2}$$

assuming that three of the four wrenches are indepen-

dent. The wrench $\hat{\mathbf{w}}_i$ and the convex $C_i^\lessgtr$ correspond, respectively, to a point contact and a range of point contacts on edge e_i.

Equation (2) can be rewritten as either of the following:

$$
\begin{aligned}
(\alpha_1 \hat{\mathbf{w}}_1) &= -(\alpha_2 \hat{\mathbf{w}}_2 + \alpha_3 \hat{\mathbf{w}}_3 + \alpha_4 \hat{\mathbf{w}}_4), \\
(\alpha_1 \hat{\mathbf{w}}_1 + \alpha_2 \hat{\mathbf{w}}_2) &= -(\alpha_3 \hat{\mathbf{w}}_3 + \alpha_4 \hat{\mathbf{w}}_4),
\end{aligned}
\tag{3}
$$

which imply the following necessary conditions:

$$
\begin{aligned}
(C_1^\lessgtr) \cap -(C_2^\lessgtr \cup C_3^\lessgtr \cup C_4^\lessgtr) &\neq \emptyset, \\
(C_1^\lessgtr \cup C_2^\lessgtr) \cap -(C_3^\lessgtr \cup C_4^\lessgtr) &\neq \emptyset.
\end{aligned}
\tag{4}
$$

By permutating the indexes, we get five other necessary conditions:

$$
\begin{aligned}
(C_2^\lessgtr) \cap -(C_3^\lessgtr \cup C_4^\lessgtr \cup C_1^\lessgtr) &\neq \emptyset, \\
(C_3^\lessgtr) \cap -(C_4^\lessgtr \cup C_1^\lessgtr \cup C_2^\lessgtr) &\neq \emptyset, \\
(C_4^\lessgtr) \cap -(C_1^\lessgtr \cup C_2^\lessgtr \cup C_3^\lessgtr) &\neq \emptyset, \\
(C_1^\lessgtr \cup C_3^\lessgtr) \cap -(C_2^\lessgtr \cup C_4^\lessgtr) &\neq \emptyset, \\
(C_2^\lessgtr \cup C_3^\lessgtr) \cap -(C_4^\lessgtr \cup C_1^\lessgtr) &\neq \emptyset.
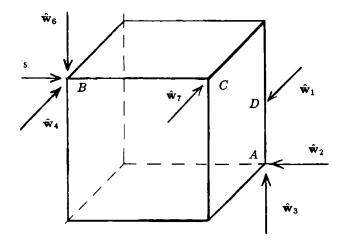\end{aligned}
\tag{5}
$$

The intersection of two convexes generally gives a smaller convex, so the necessary conditions in Eq. (5) restrict edges $e_1, \ldots, e_4$ to smaller segments $s_1, \ldots, s_4$. These segments represent the independent regions of contact if their respective convexes are disjoint. In a plane, two nonparallel lines always intersect. This is why $C_1^\lessgtr \cup C_2^\lessgtr$ can be geometrically represented by point P_{12} and sector $\mathscr{C}_{12}$. The geometric representation is exact, because it captures all the strictly positive combinations of vectors in $C_1^\lessgtr$ and $C_2^\lessgtr$.

The fitting of the two-sided cone $C^\times(I, \pm\mathscr{C})$ between parallelograms Π_{12} and Π_{34} captures the operation $(C_1^\lessgtr \cup C_2^\lessgtr) \cap -(C_3^\lessgtr \cup C_4^\lessgtr)$. The enumeration of the regions R_i and R_j, the pairwise intersection of these regions, $R_i \cap R_j$, and the computation of the locally optimum vertex I are just a geometric search of the optimum set of four disjoint convexes that satisfy the necessary condition

$$(C_1^\lessgtr \cup C_2^\lessgtr) \cap -(C_3^\lessgtr \cup C_4^\lessgtr) \neq \emptyset. \tag{6}$$

The geometric construction makes explicit the location of the points of contact. If the four convexes found from Eq. (6) are disjoint and three of the four convexes are linearly independent, then the other six necessary

be applied by pushing through vertices A and B only. By adding $\hat{\mathbf{w}}_7$ at vertex C and placing the point of contact of $\hat{\mathbf{w}}_1$ at point D, we can create torques in both directions about AB and make grasp G force-closure.

Constructing the seven independent regions of contact is more expensive and harder. A convex region for a point of contact on a planar face corresponds to a wrench convex in the 6D wrench space. The problem is to find seven disjoint wrench convexes in this 6D space such that any seven-tuple of wrenches from these seven convexes is force-closure. Due to the convexity of the domain, the problem is transformed into finding seven disjoint wrench convexes such that six of the convexes are linearly independent and satisfy one of the following equations:

conditions in Eqs. (4) and (5) are also satisfied. This explains why we need to consider only one pairing of edges e_1, e_2 against e_3, e_4, instead of all three. Equation (6) plus the condition that the convexes are disjoint and that they linearly span the space is a necessary and sufficient condition for force-closure grasps with independent regions of contact.

$$(C_1^{\leqslant}) \cap -(C_2^{\leqslant} \cup C_3^{\leqslant} \cup C_4^{\leqslant} \cup C_5^{\leqslant} \cup C_6^{\leqslant} \cup C_7^{\leqslant}) \neq \emptyset,$$
$$(C_1^{\leqslant} \cup C_2^{\leqslant}) \cap -(C_3^{\leqslant} \cup C_4^{\leqslant} \cup C_5^{\leqslant} \cup C_6^{\leqslant} \cup C_7^{\leqslant}) \neq \emptyset, \quad (7)$$
$$(C_1^{\leqslant} \cup C_2^{\leqslant} \cup C_3^{\leqslant}) \cap -(C_4^{\leqslant} \cup C_5^{\leqslant} \cup C_6^{\leqslant} \cup C_7^{\leqslant}) \neq \emptyset.$$

There are $\binom{7}{1}$, $\binom{7}{2}$, and $\binom{7}{3}$ equations, respectively, similar to the three in Eq. (7). Note that nonparallel lines in space do not generally intersect, so there is no geometric construction for 3D grasps with seven frictionless point contacts, as there is for 2D grasps with four frictionless point contacts.

Let d be the dimension of the space, and let c be the number of required contacts ($d = 6$ and $c = 7$ for frictionless point contacts in 3D). The c wrench convexes, corresponding to c regions of contact on the object, generally overlap in the d-dimensional wrench space. The number of possible intersections among c convexes is $O(2^c)$. For each such intersection, we locally trade off between the intersecting convexes to get a set of c disjoint subconvexes that have largest minimum independent region of contact. Then, we check for vector-closure on a representative c-tuple of vectors taken from these disjoint subconvexes. In this way, we enumerate all c-tuples of disjoint subconvexes from the c convexes, and check each for vector-closure.

Assuming that convexes are approximated by circular cones, we can intersect the two convexes in constant time. Intersecting cones are traded against each other by changing their respective sector. The local tradeoff is done in $O(c)$ time because the area of the region of contact is directly proportional to the sector of the cone. Checking vector-closure of c vectors in

5.5. 3D Grasps with Seven Frictionless Contacts

Without friction, we need at least seven frictionless point contacts instead of two soft-finger contacts or three hard-finger contacts. Figure 11 shows a force-closure grasp G on a cube with seven frictionless point contacts. The wrenches are:

$$\hat{\mathbf{w}}_1 = (1 \quad 0 \quad 0 \quad 0 \quad 0.5 \quad -1),$$
$$\hat{\mathbf{w}}_2 = (0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0),$$
$$\hat{\mathbf{w}}_3 = (0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0),$$
$$\hat{\mathbf{w}}_4 = (-1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0),$$
$$\hat{\mathbf{w}}_5 = (0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 1),$$
$$\hat{\mathbf{w}}_6 = (0 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0),$$
$$\hat{\mathbf{w}}_7 = (-1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1).$$

One can verify that $2\hat{\mathbf{w}}_1 + \hat{\mathbf{w}}_2 + \hat{\mathbf{w}}_3 + \hat{\mathbf{w}}_4 + \hat{\mathbf{w}}_5 + \hat{\mathbf{w}}_6 + \hat{\mathbf{w}}_7 = \hat{\mathbf{0}}$. If we remove $\hat{\mathbf{w}}_7$ and move the point of contact of $\hat{\mathbf{w}}_1$ to vertex A, then grasp G is similar to a grasp with two opposing hard-finger contacts pressing at A and B. Note that no torque about the diagonal AB can

d-dimensional space is equivalent to doing Gaussian elimination on a $c \times d$ matrix, and so costs $O(cd^2)$ time. Hence, constructing c optimal disjoint subconvexes that span the whole space costs $O(c2^c)$ time. This $O(c2^c)$ complexity makes the construction for independent regions of contact more expensive for frictionless grasps ($c = 7$) than for grasps with friction ($c = 2$ or 3).

Redundant contacts do not change the force-closure property of the grasp and so can be placed anywhere on the object. Optimal grasps on $n > c$ given regions can be found by finding optimal grasps for all $\binom{n}{c}$ c-tuples of the grasping regions, and so costs $O(n^c c2^c)$ time.

COMPLEXITY 1 *Let G be a grasp with n fingertips, requiring c minimum contacts ($n > c$).*

1. Analyzing whether grasp G at n given contacts is force-closure or not costs $O(n)$ time.

2. Synthesizing the n independent regions of contacts of G costs $O(n^c c2^c)$ time.

The construction of the independent contact regions is tranformed into the problem of finding disjoint convexes satisfying necessary conditions like Eq. (7). This transformation depends on the convexity property and on the correspondence between vectors in the convex and points of contact on the object. Formally, any affine combination of two point contacts P_1 and P_2 inside a contact region must be a point contact P inside the same region:

$$\alpha \begin{bmatrix} s_1 \\ s_{10} \end{bmatrix} + (1 - \alpha) \begin{bmatrix} s_2 \\ s_{20} \end{bmatrix} = \begin{bmatrix} s \\ s_0 \end{bmatrix}, \qquad 0 \le \alpha \le 1. \quad (8)$$

A frictionless point contact is represented by a pure force going through the point of contact and normal to the surface. A pure force is a line vector, so the dot product of the upper and lower parts of the spatial vector must be zero:

$$s_1 \cdot s_{10} = 0, \quad s_2 \cdot s_{20} = 0, \quad s \cdot s_0 = 0. \quad (9)$$

Substituting s and s_0 from Eq. (8) into the third equation of Eq. (9), we deduce that the following must hold:

$$s_1 \cdot s_{20} + s_{10} \cdot s_2 = 0. \quad (10)$$

This equation expresses the condition that the two lines of force at points P_1 and P_2 must intersect or be parallel. If we extrapolate this condition to other points of the contact region, the convexity property and the correspondence between vectors and contact points imply that the region of contact must (1) be either flat or spherical, and (2) have a convex boundary. This explains why the construction of the contact regions is so simple for polygonal and polyhedral objects. This suggests that frictionless grasps on curved regions with constant curvature can be synthesized similarly to the construction sketched here. Only friction can relax this constant-curvature condition.

6. Conclusion

Finding places to put the fingertips is formalized into finding independent regions of contact such that the grasp is force-closure. Constructing force-closure grasps is definitely a geometric problem, and one that is very simple for two point contacts with friction in 2D or two soft-finger contacts in 3D. For two such contacts, the contact regions are back-to-back or face-to-face.

We have shown how to construct the independent regions of contact for grasps on polygonal and polyhedral objects. We have analyzed why these contact regions are harder to construct for arbitrary curved objects, and sketched how such regions can be directly constructed from local regions of constant curvature so that incremental search as in Asada (1979) is not needed. The current synthesis can be extended to handle grasps with edge and face contacts. A more interesting extension is the synthesis of grasps that use the structural restraint from many contacts on different links of a same finger (Trinkle, Abel, and Paul 1987). Finding a formal framework for these structural grasps will give us deep insight into the power grasps found in humans (Cutkosky and Wright 1986).

Acknowledgment

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