12. (15 points) Air is pumped into a spherical balloon at a rate of 8 cm^3/s. When the radius of the balloon is 8 cm, at what rate is the surface area of the balloon growing? (Hint: For a sphere, volume is $V = \frac{4}{3}\pi r^3$ and surface area is $A = 4\pi r^2$, where r is the radius of the sphere. Use these equations to related volume to surface area.)

Since $V = \frac{4}{3}\pi r^3$, we have $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

Given $\frac{dV}{dt} = 8$ & $r = 8$, $8 = 4\pi \cdot 8^2 \cdot \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{8}{4\pi \cdot 8^2} = \frac{1}{32\pi}$

Also, since $A = 4\pi r^2$, $\frac{dA}{dt} = 4\pi \cdot 2r \frac{dr}{dt}$

Substitute $r = 8$ & $\frac{dr}{dt} = \frac{1}{32\pi}$ & we get

$\frac{dA}{dt} = 4\pi \cdot 2 \cdot 8 \cdot \frac{1}{32\pi} = 2$

The surface area of the baloon is growing at the rate of $2 cm^2/sec$. //

Alternative solution

Solve $V = \frac{4}{3}\pi r^3$ for r.

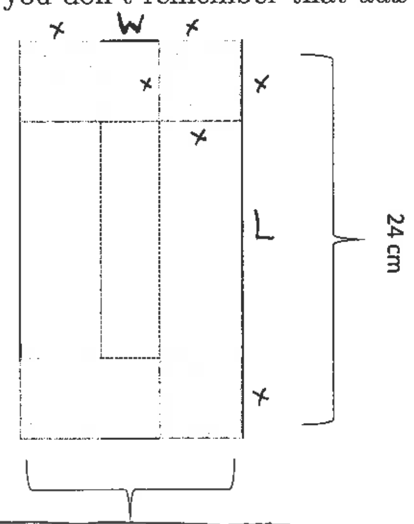
$r^3 = \frac{3V}{4\pi} \Rightarrow r = \left(\frac{3V}{4\pi}\right)^{1/3}$. Substitute this into the formula for the surface area to obtain

$A = 4\pi \left(\frac{3V}{4\pi}\right)^{2/3}$. Therefore $\frac{dA}{dt} = 4\pi \cdot \frac{2}{3}\left(\frac{3V}{4\pi}\right)^{-1/3} \frac{3}{4\pi} \frac{dV}{dt}$

Use $\frac{dV}{dt} = 8$ & $V = \frac{4}{3}\pi 8^3$, so

$\frac{dA}{dt} = 4\pi \cdot \frac{2}{3}\left(\frac{3}{4\pi} \cdot \underbrace{\frac{4}{3}\pi \cdot 8^3}_{V}\right)^{-1/3} \cdot \frac{3}{4\pi} \cdot 8 = 2 \ cm^2/sec$

13. (15 points) You have a rectangular piece of cardboard measuring 9 centimeters by 24 centimeters. You want to make a topless box by cutting four identical squares out of the cardboard, one from each corner, and then folding the four sides up. What is the maximum possible volume of the topless box? (Hint: Some of the algebra can be messy if you don't remember that $adx^2 + bdx + cd = d(ax^2 + bx + c)$.)



Let x be the side length of the squares which are being removed.

The box will have dimensions L by W by x.

Volume $= V$. The volume of the box is

$$V = LWx.$$

Since $L + 2x = 24$, $L = 24 - 2x$.

Since $W + 2x = 9$, $W = 9 - 2x$.

Therefore, $V(x) = (24-2x)(9-2x)x$

$= 2(12-x)(9-2x)x$

$= 2(108 - 9x - 24x + 2x^2)x$

$= 2(108 - 33x + 2x^2)x$

$= 2(108x - 33x^2 + 2x^3)$.

Since we need all our dimensions to be non-negative, we need $x \geq 0$, $24 - 2x \geq 0$, and $9 - 2x \geq 0$

$\Rightarrow 24 \geq 2x \Rightarrow 12 \geq x$

$\Rightarrow 9 \geq 2x \Rightarrow 9/2 \geq x$.

So the only relevant values of x are $[0, 9/2]$.

By the extreme value theorem, the maximum volume must occur either at an endpoint, or at a critical point. To find the critical points, we find where $V'(x) = 0$:

$V'(x) = 2(108 - 66x + 6x^2)$

$= 12(18 - 11x + x^2)$, by the Power Rule.

$= 12(9-x)(2-x)$.

The critical points are at $x = 9$ and $x = 2$, but 9 is not in $[0, 9/2]$, so we can ignore it.

9

Finally, we compute the volume at our critical points and at the endpoints:

$V(0) = (24-2(0))(9-2(0))(0) = 0$.

$V(2) = (24-2(2))(9-2(2))(2)$

$= 20(5)(2)$

$= 200$.

$V(9/2) = (24-2(9/2))(9-2(9/2))(9/2) = 0$.

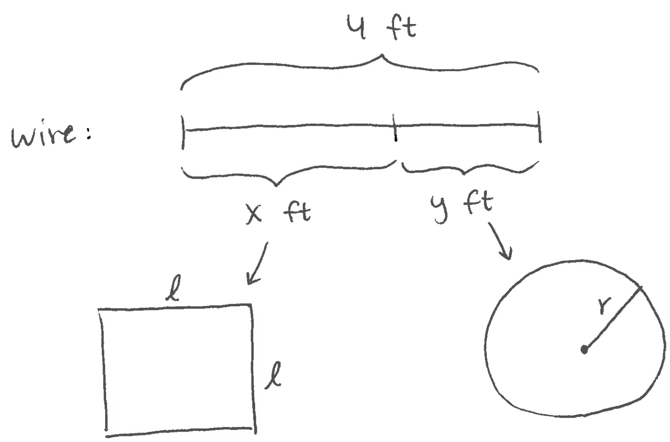
Since the largest value is 200, this is the maximum volume. □

Long answer questions

For each long answer problem, you must give a full explanation and justification of your answer to receive credit. A list of computations is not sufficient to gain credit!

11. (15 points) Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area (i.e. the sum of the area of the circle and the area of the square)?

Solution: Let's start by drawing a picture and labeling the quantities involved:



There are many ways to proceed in this problem, depending on which quantity you decide will be the independent variable. I will start by defining the following

$$
\begin{aligned}
x &= \text{ length of wire used for square}\\
y &= \text{ length of wire used for circle}\\
c &= \text{ circumference of the circle}\\
r &= \text{ radius of the circle}\\
\ell &= \text{ length of one side of the square}\\
A_s &= \text{ area of square}\\
A_c &= \text{ area of circle}\\
A_{tot} &= A_s + A_c = \text{ total area of both the square and circle}
\end{aligned}
$$

The first constraint given by the problem is that the perimeter of the square and the circle must sum to four, $x + y = 4$. So we can eliminate x or y from the problem, as we so desire. Next, using simple geometry we have the following relations between the quantities above:

$$\ell = \frac{x}{4} \qquad y = c = 2\pi r$$

$$A_s = \ell^2 = \left(\frac{x}{4}\right)^2 = \left(\frac{4-y}{4}\right)^2 = \left(\frac{4-c}{4}\right)^2 = \left(\frac{4-2\pi r}{4}\right)^2 = \left(\frac{2-\pi r}{2}\right)^2$$

$$A_c = \pi r^2 = \pi r^2 = \pi\left(\frac{c}{2\pi}\right)^2 = \pi\left(\frac{y}{2\pi}\right)^2 = \pi\left(\frac{4-x}{2\pi}\right)^2 = \pi\left(\frac{4-4\ell}{2\pi}\right)^2 = \pi\left(\frac{2-2\ell}{\pi}\right)^2$$

Now we use this to derive five different functions for A_{tot}, one for each of the five choices of independent variable above, together with the range of values over which A_{tot} should be maximized . In addition, we include the first and second derivatives and any critical numbers.

Independent variable is x:

$$A_{tot}(x) = A_s + A_c = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{4-x}{2\pi}\right)^2 \qquad 0 \le x \le 4$$

$$A'_{tot}(x) = \frac{x}{8} + \frac{x-4}{2\pi}$$

$$A''_{tot}(x) = \frac{1}{8} + \frac{1}{2\pi}$$

only critical number: $x = \dfrac{16}{\pi+4}$

Independent variable is y:

$$A_{tot}(y) = A_s + A_c = \left(\frac{4-y}{4}\right)^2 + \pi\left(\frac{y}{2\pi}\right)^2 \qquad 0 \le y \le 4$$

$$A'_{tot}(y) = \frac{y-4}{8} + \frac{y}{2\pi}$$

$$A''_{tot}(y) = \frac{1}{8} + \frac{1}{2\pi}$$

only critical number: $y = \dfrac{4\pi}{\pi+4}$

Independent variable is c:

$$A_{tot}(c) = A_s + A_c = \left(\frac{4-c}{4}\right)^2 + \pi\left(\frac{c}{2\pi}\right)^2 \qquad 0 \le c \le 4$$

$$A'_{tot}(c) = \frac{c-4}{8} + \frac{c}{2\pi}$$

$$A''_{tot}(c) = \frac{1}{8} + \frac{1}{2\pi}$$

only critical number: $c = \dfrac{4\pi}{\pi+4}$

Independent variable is r:

$$A_{tot}(r) = A_s + A_c = \left(\frac{2-\pi r}{2}\right)^2 + \pi r^2 \qquad 0 \le r \le \frac{2}{\pi}$$

$$A'_{tot}(t) = \frac{\pi^2 r - 2\pi}{2} + 2\pi r$$

$$A''_{tot}(t) = \frac{\pi^2}{2} + 2\pi$$

only critical number: $r = \dfrac{2}{\pi+4}$

Independent variable is ℓ:

$$A_{tot}(\ell) = A_s + A_c = \ell^2 + \pi\left(\frac{2-2\ell}{\pi}\right)^2 \qquad 0 \le \ell \le 1$$

$$A'_{tot}(\ell) = 2\ell + \frac{8\ell-8}{\pi}$$

$$A''_{tot}(\ell) = 2 + \frac{8}{\pi}$$

only critical number: $\ell = \dfrac{4}{\pi+4}$

In each case we note that the second derivative is always positive, so by the second derivative test, any critical point will be a local minimum. One could also have noted that each of the functions above is a concave-up parabola, so its unique critical point must be a local minimum. In either case, it cannot be an absolute maximum at the indicated critical point.

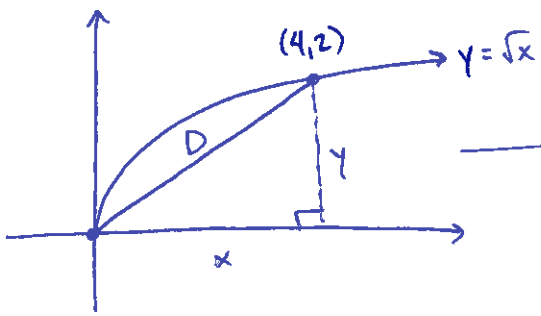
So, by the Extreme Value Theorem or the Closed Interval Method, the maximum area must occur at the one of the endpoints of the specified interval. Taking the case of

$A_{tot}(r)$ as an example, we find

$$A_{tot}(0) = 1 \qquad A_{tot}\left(\frac{2}{\pi}\right) = \pi\frac{4}{\pi^2} = \frac{4}{\pi}$$

Thus we conclude that the area enclosed is maximized when $r = 2/\pi$ and that area is $4/\pi$. In terms of the problem, this corresponds to the situation when all the wire is used for the circle.

14. (15 points) A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?



→ distance formula / Pythagorean theorem

$$D^2 = x^2 + y^2$$
$$= x^2 + (\sqrt{x})^2$$
$$= x^2 + x.$$

(+5)

Implicit differentiation → $2D \cdot \frac{dD}{dt} = 2x \cdot \frac{dx}{dt} + \frac{dx}{dt}$

$$\Rightarrow \frac{dD}{dt} = \frac{(2x+1) \cdot \frac{dx}{dt}}{2D}$$

(+7)

What is D when $x = 4$? $D^2 = (4)^2 + 4 = 20 \Rightarrow$ $\boxed{D = \sqrt{20}}$

Therefore, $\frac{dD}{dt} = \frac{(2 \cdot 4 + 1) \cdot 3}{2\sqrt{20}} = \frac{9 \cdot 3}{2\sqrt{20}} = \boxed{\frac{27}{2\sqrt{20}} \text{ cm/sec}}$

(+3)

15. (20 points) Let

$$f(x) = \frac{2x^2 - 3x}{x - 2}.$$

Then, $f'(x) = \dfrac{2(x^2 - 4x + 3)}{(x-2)^2}$ and $f''(x) = \dfrac{4}{(x-2)^3}$.

(a) If there are any, find the horizontal and vertical asymptote(s) of f.

(b) Determine the critical point(s) of f. If they exist, classify each critical point as a local maximum, a local minimum, or neither.

(c) Determine the intervals on which f is increasing/decreasing.

(d) Find the inflection point(s) of f, or show there are no inflection points.

(e) Determine the intervals on which f is concave up/down.

(f) Accurately graph the function $f(x)$ on the axes provided on page 13. Label all points and asymptotes found in the previous parts of the question.

Solution: (a) To find horizontal asymptotes, we compute two limits. First, we take the limit as x approached infinity:

$$\lim_{x\to\infty} \frac{2x^2 - 3x}{x - 2} = \lim_{x\to\infty} \frac{x(2x-3}{x(1-2/x)} = \lim_{x\to\infty} \frac{2x-3}{1-2/x}.$$

Since $2/x$ tends to zero as x tends to infinity, the denominator tends to 1 as x approaches infinity. As the numerator tends to infinity, the quotient also tends to infinity. We take the second limit as x approaches negative infinity:

$$\lim_{x\to-\infty} \frac{2x^2 - 3x}{x - 2} = \lim_{x\to-\infty} \frac{x(2x-3}{x(1-2/x)} = \lim_{x\to-\infty} \frac{2x-3}{1-2/x}.$$

Since $2/x$ tends to zero as x tends to negative infinity, the denominator tends to 1 as x approaches infinity. As the numerator tends to negative infinity as x approaches negative infinity, the quotient tends to negative infinity as well. Consequently, there are no horizontal asymptotes.

To find vertical asymptotes, we look for values $x = a$ where one or both of the one-sided limits of $f(x)$ as x approaches a become unbounded. Our candidates for values of a are zeros of the denominator, as these are places where the function *potentially* tends to $\pm\infty$. Setting the denominator equal to zero gives $(x-2) = 0$ or $x = 2$. We check the left and right-handed limits. We claim that

$$\lim_{x\to 2^-} \frac{2x^2 - 3x}{x - 2} = -\infty.$$

This is true because as x approaches two from below, the numerator tends to 6 while the numerator tends to zero from below. Similarly,

$$\lim_{x\to 2^+} \frac{2x^2 - 3x}{x - 2} = \infty,$$

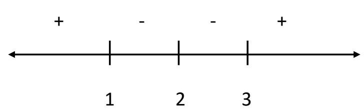
as the numerator still tends to 6 but the denominator tends to zero from above.

(b) To isolate the critical points, we first look for x in the domain of f where $f'(x) = 0$ or where $f'(x)$ is not defined. Setting $f'(x) = 0$ yields

$$f'(x) = \frac{2(x^2 - 4x + 3)}{(x-2)^2} = \frac{2(x-3)(x-1)}{(x-2)^2} = 0,$$

and we see that $f'(3) = f'(1) = 0$. While $x = 2$ is a location where $f'(x)$ is not defined, it is not a critical number because it is not in the domain of f. As $f(3) = (18 - 9)/(3 - 2) = 9$ and $f(1) = (2 - 3)(1 - 2) = 1$, our critical points are $(3, 9)$ and $(1, 1)$. We know two ways to classify the critical points that apply in this problem - the first and second derivative tests. For completeness, we'll present both, but either is sufficient.

To use the first derivative test, we plug in points to determine the sign of the derivative in the intervals $(-\infty, 1)$, $(1, 2)$, $(2, 3)$ and $(3, \infty)$. As $f'(0) = 6/4 > 0, f'(3/2) = 2(-3/2)(1/2)/(1/2)^2 < 0, f'(5/2) = 2(-1/2)(3/2)/(1/2)^2 < 0$, and $f'(4) = 12/8 > 0$, we have that $f'(x) > 0$ on $(-\infty, 1) \cup (3, \infty)$ and $f'(x) < 0)$ on $(1, 2) \cup (2, 3)$ (see figure). Consequently, the first derivative test says that $(1, 1)$ is a local maximum and $(3, 9)$ is a local minimum.



To use the second derivative test, we plug $x = 1$ and $x = 3$ into $f''(x)$, yielding $f''(1) = 4/(-1)^3 = -4$ and $f''(3) = 4/(2)^2 = 1$. So, the function is concave down at $(1, 1)$ and hence the point is a local maximum and the function is concave up at $(3, 9)$ and hence the point is a local minimum.

(c) If we used the first derivative test in part (b), much of the work is done and we conclude that f is increasing on $(-\infty, 1) \cup (3, \infty)$ and decreasing on $(1, 2) \cup (2, 3)$.

(d) To find potential inflection points, we look for x in the domain where $f''(x) = 0$. Since the numerator of f'' is constant, we cannot find any zeros in the domain. Therefore, there are no inflection points.

(e) To determine concavity, we test points in the two intervals that make up our domain: $(infty, 2)$ and $(2, \infty)$. As $f''(0) = 4/(-2)^3 = -1/2 < 0$ and $f''(3) = 4/1^3 = 4 > 0$, we see that f is concave down on $(-\infty, 2)$ and is concave up on $(2, \infty)$.

Here is a set of axes where you may sketch the curve for problem 15.