Long answer questions

For each long answer problem, you must give a full explanation and justification of your answer to receive credit. A list of computations is not sufficient to gain credit!

11. (15 points) Let \( f(x) = \frac{x - 10}{x - 5} \). Find \( f'(x) \) and \( f''(x) \). Then find all the points where \( f'(x) = f''(x) \). Remember you must justify your answers to receive credit. Solution:

First, we find \( f'(x) \) using differentiation rules:

\[
\frac{d}{dx} f(x) = \frac{d}{dx} \frac{x - 10}{x - 5}.
\]

By the quotient rule this is

\[
\frac{\left( \frac{d}{dx} (x - 10) \right) (x - 5) - (x - 10) \frac{d}{dx} (x - 5)}{(x - 5)^2}.
\]

Since, by the power and sum rules, \( \frac{d}{dx} (x - 10) = \frac{d}{dx} (x - 5) = 1 \), this simplifies to

\[
\frac{1(x - 5) - (x - 10)1}{(x - 5)^2} = \frac{x - 5 - x + 10}{(x - 5)^2} = \frac{5}{(x - 5)^2}.
\]

Second, we’ll find \( f''(x) \) using differentiation rules. While we could use the quotient rule again, we’ll instead write \( f'(x) = 5(x - 5)^{-2} \) and use the constant multiple, power, and chain rules to find,

\[
f''(x) = 5(-2)(x - 5)^{-2-1} = -10(x - 5)^{-3} = \frac{-10}{(x - 5)^3}.
\]

To find the values of \( x \) where \( f'(x) = f''(x) \), we set the two equal and algebraically simplify:

\[
\frac{5}{(x - 5)^2} = \frac{-10}{(x - 5)^3}
\]

\[
5(x - 5) = -10 \text{ (multiply both sides by } (x - 5)^3) \]

\[
5x = -10 + 25 = 15
\]

\[
x = 3
\]

To find the \( y \) value associated to this point, we use the definition of \( f \),

\[
f(3) = \frac{3 - 10}{3 - 5} = \frac{-7}{-2} = \frac{7}{2},
\]

so the only point where the two are equal is \((3, \frac{7}{2})\).
12. (10 points) Compute the second derivatives of each of the following functions:

(a) \( f(x) = \sin(e^x) \).
(b) \( h(x) = (1 + 2x^2)^{10} \).

Remember you must justify your answers to receive credit.

a. For the first derivative, we must use the chain rule:
   \[ f'(g(x)) = f'(g(x)) \cdot g'(x) \]
   In combination with the special trig derivatives:
   \[ f(x) = \sin(e^x) \quad \text{Let } F(x) = \sin(x), \quad G(x) = e^x \]
   Then \[ F'(x) = \cos(e^x) \cdot e^x \]
   For the second derivative, we must use the product rule:
   \[ f''(x) = f'(x)g'(x) + f(x)g''(x) \]
   Let \( F(x) = e^x \), \( G(x) = \cos(e^x) \). Then
   \[ F''(x) = e^x \cos(e^x) + e^x (\cos(e^x))' \]
   We use the chain rule for \((\cos(e^x))'\) similar to above:
   \[ F''(x) = e^x \cos(e^x) - (e^x)^2 \sin(e^x) \]

b. For the first derivative, we must use the chain rule:
   Let \( F(x) = x^{10}, \quad G(x) = 1 + 2x^2 \). Then
   \[ F'(x) = 10(1 + 2x^2)^9 \cdot 4x \]
   For the second derivative, we must use the product rule:
   Let \( F(x) = 10(1 + 2x^2)^9, \quad G(x) = 4x \). Then
   \[ F''(x) = (10(1 + 2x^2)^9)' \cdot 4x + (10(1 + 2x^2)^9) \cdot 4 \]
   To find \( F''(x) \), we need to use the chain rule, with the outer function being \( 10x^9 \) and the inner being \( 1 + 2x^2 \) (similar to above)
   Then
   \[ F''(x) = (90(1 + 2x^2)^8) \cdot 4x \cdot 4x + 40(1 + 2x^2)^9 \]
   \[ = 1440x^2 (1 + 2x^2)^8 + 40(1 + 2x^2)^9 \]
13. (20 points) Let \( g(y) = \frac{y}{y-3} \). Remember you must justify your answers to receive credit.

a) What is the domain of \( g(y) \)?

Since division by 0 is not allowed, the function \( g(y) \) is not defined when \( y-3=0 \), or equivalently when \( y=3 \).

Therefore the domain of \( g \) is \( \{ y \mid y \neq 3 \} \), or in interval notation,
\[ (-\infty, 3) \cup (3, \infty) \]

b) Where is \( g \) continuous? Classify any discontinuities you find.

Since \( g \) is a rational function, we know that it is continuous wherever it is defined. Therefore, it is continuous on its domain, \( (-\infty, 3) \cup (3, \infty) \).

The only discontinuity is at \( y=3 \).

Since we have
\[ \lim_{y \to 3^+} \frac{y}{y-3} = \infty \quad \text{and} \quad \lim_{y \to 3^-} \frac{y}{y-3} = -\infty, \]
it has an infinite discontinuity at \( y=3 \).

(This problem is continued on the next page.)
c) What is the derivative of \( g(y) \)?

\[
\frac{d}{dy} g(y) = \frac{(y)' \cdot (y-3) - y \cdot (y-3)'}{(y-3)^2}
\]

\[
= \frac{1 \cdot (y-3) - y \cdot 1}{(y-3)^2}
\]

\[
= -\frac{3}{(y-3)^2}.
\]

d) Find the equation of the tangent line to \( z = g(y) \) at the point where \( y = 4 \) and \( z = 4 \).

The slope of the tangent line at \( y = 4 \) is \( g'(4) \).

Therefore \( m = g'(4) = -\frac{3}{(4-3)^2} = -3 \).

Using the point-slope form of the equation of a line, with the given point \((4, 4)\), we obtain an equation of the tangent line as

\[
z - 4 = -3(y - 4).
\]

*Alternative solution*: The slope-intercept form gives \( z = -3y + b \). Substituting \( y = 4 + z = 4 \), we get \( 4 = -3 \cdot 4 + b \), or \( b = 4 + 12 = 16 \).

Therefore, the equation of the tangent line is \( z = -3y + 16 \).
14. (15 points) Let \( h(s) = \cos(s) + s^4 - 3s^2 + 5 \). Remember you must justify your answers to receive credit.

a) State the definitions of even and odd for functions.

A function \( f \) is called even if \( f(-x) = f(x) \) for every number \( x \).

A function \( f \) is called odd if \( f(-x) = -f(x) \) for every number \( x \).

b) Find \( h'(s) \).

Using the rule \( \frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)] \), as well as the special derivative \( \frac{d}{ds} [\cos(s)] = -\sin(s) \) and power rule, we have

\[
h'(s) = \frac{d}{ds} [\cos(s)] + \frac{d}{ds} [s^4] - 3 \frac{d}{ds} [s^2] + \frac{d}{ds} [5] \\
= -\sin(s) + 4s^3 - 6s + 0 \\
= -\sin(s) + 4s^3 - 6s
\]

(This problem is continued on the next page.)
c) Is $h(s)$ even, odd, both, or neither? Is $h'(s)$ even, odd, both, or neither?

First note that $\cos(\theta)$ is an even function. This is because the point $(x, y)$ on the unit circle corresponding to the angle $\theta$ has the same $x$-value as the point corresponding to the angle $-\theta$:

\[
\begin{align*}
\cos \theta &= x \\
\cos (-\theta) &= x
\end{align*}
\]

By similar reasoning, $\sin \theta$ is an odd function, i.e. $\sin(-\theta) = -\sin(\theta)$. So for any number $s$

\[
\begin{align*}
h(-s) &= \cos(-s) + (-s)^4 - 3(-s)^2 + 5 \\
&= \cos(s) + s^4 - 3s^2 + 5 \\
&= h(s)
\end{align*}
\]

and $h(s)$ is even. Similarly for any number $s$

\[
\begin{align*}
h'(-s) &= -\sin(-s) + 4(-s)^3 - 6(-s) \\
&= (-\sin(s)) + 4s^3 + 6s \\
&= (-\sin(s) + 4s^3 - 6s) \\
&= -h'(s)
\end{align*}
\]

and $h'(s)$ is odd.
15. (10 points) Let
\[ f(x) = x^3 - 6x^2 - 15x + 128. \]
Find all the points on the graph \( y = f(x) \) where the tangent line is horizontal. Remember you must justify your answers to receive credit.

+3 The tangent line to a graph is horizontal precisely when \( f'(c) = 0 \).

+1 Using the Power Rule, we compute \( f''(x) = 3x^2 - 12x - 15 \).

+4 And then by setting \( f'(x) = 0 \), we solve for \( x \), obtaining:

\[
\begin{align*}
f''(x) &= 0 \\
3(x^2 - 4x - 5) &= 0 \\
3(x - 5)(x + 1) &= 0 \\
x &= 5, x = -1
\end{align*}
\]

+1 To find the points on the graph, we then evaluate \( f(5) \) and \( f(-1) \) to find the \( y \)-values:

\[
f(5) = (5)^3 - 6(5)^2 - 15(5) + 128 = 125 - 150 - 75 + 128 = 28 \\
\Rightarrow (5, 28)
\]

\[
f(-1) = (-1)^3 - 6(-1)^2 - 15(-1) + 128 = -1 - 6 + 15 + 128 = 136 \\
\Rightarrow (-1, 136)
\]
16. (10 points) Let \( h(x) \) be defined by

\[
h(x) = \begin{cases} 
  x^2 \sin \left( \frac{x^3 - 4x + 3}{x} \right) & x \neq 0, \\
  c & x = 0,
\end{cases}
\]

where \( c \) is a constant. For which value of \( c \) is \( h(x) \) continuous at \( x = 0 \)? Hint: Recall that \(-1 \leq \sin(z) \leq 1\) for all \( z \). Remember you must justify your answers to receive credit.

\[-1 \leq \sin \left( \frac{x^3 - 4x + 3}{x} \right) \leq 1 \quad \text{for all } x \neq 0.\]

Multiplying by \( x^2 \) gives us

\[-x^2 \leq x^2 \sin \left( \frac{x^3 - 4x + 3}{x} \right) \leq x^2 \quad \text{for all } x \neq 0.\]

Taking limits as \( x \) approaches 0, we get

\[
\lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \sin \left( \frac{x^3 - 4x + 3}{x} \right) \leq \lim_{x \to 0} x^2.
\]

Since \( \lim_{x \to 0} -x^2 = 0 \) and \( \lim_{x \to 0} x^2 = 0 \), by the squeeze theorem, we get

\[
\lim_{x \to 0} x^2 \sin \left( \frac{x^3 - 4x + 3}{x} \right) = 0,
\]

so

\[
\lim_{x \to 0} h(x) = 0.
\]

For \( h(x) \) to be continuous at \( x = 0 \), we need \( \lim_{x \to 0} h(x) = h(0) \).

Since \( \lim_{x \to 0} h(x) = 0 \), and \( h(0) = c \), setting \( c = 0 \) makes \( h(x) \) continuous at \( x = 0 \).