Long answer questions

For each long answer problem, you must give a full explanation and justification of your answer to receive credit. A list of computations is not sufficient to gain credit!

11. (15 points) Let $f(x) = \dfrac{x-10}{x-5}$. Find $f'(x)$ and $f''(x)$. Then find all the points where $f'(x) = f''(x)$. Remember you must justify your answers to receive credit. **Solution:**

First, we find $f'(x)$ using differentiation rules:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\frac{x-10}{x-15}.$$

By the quotient rule this is

$$\frac{(\frac{d}{dx}(x-10))(x-5) - (x-10)\frac{d}{dx}(x-5)}{(x-5)^2}.$$

Since, by the power and sum rules, $\frac{d}{dx}(x-10) = \frac{d}{dx}(x-5) = 1$, this simplifies to

$$\frac{1(x-5)-(x-10)1}{(x-5)^2} = \frac{x-5-x+10}{(x-5)^2} = \frac{5}{(x-5)^2}.$$

Second, we'll find $f''(x)$ using differentiation rules. While we could use the quotient rule again, we'll instead write $f'(x) = 5(x-5)^{-2}$ and use the constant multiple, power, and chain rules to find,

$$f''(x) = 5(-2)(x-5)^{-2-1}1 = -10(x-5)^{-3} = \frac{-10}{(x-5)^3}.$$

To find the values of x where $f'(x) = f''(x)$, we set the two equal and algebraically simplify:

$$\begin{aligned} \frac{5}{(x-5)^2} &= \frac{-10}{(x-5)^3} \\ 5(x-5) &= -10 \quad \text{(multiply both sides by } (x-5)^3\text{)} \\ 5x &= -10+25 = 15 \\ x &= 3 \end{aligned}$$

To find the y value associated to this point, we use the definition of f,

$$f(3) = \frac{3-10}{3-5} = \frac{-7}{-2} = \frac{7}{2},$$

so the only point where the two are equal is $\left(3, \frac{7}{2}\right)$.

12. (10 points) Compute the second derivatives of each of the following functions:

(a) $f(x) = \sin(e^x)$.

(b) $h(x) = (1 + 2x^2)^{10}$.

Remember you must justify your answers to receive credit.

a. For the first derivative, we must use the chain rule:

$$f(g(x)) = f'(g(x)) \cdot g'(x)$$

In combination with the special trig derivatives:

$f(x) = \sin(e^x)$. Let $F(x) = \sin(x)$, $G(x) = e^x$.

Then $\boxed{f'(x) = F'(G(x)) \cdot G'(x) = \cos(e^x) \cdot e^x}$

For the second derivative, we must use the product rule:

$$f \cdot g(x) = f'(x)g(x) + f(x)g'(x)$$

Let $F(x) = e^x$, $G(x) = \cos(e^x)$. Then

$f''(x) = e^x \cos(e^x) + e^x (\cos(e^x))'$. We use the chain rule for $(\cos(e^x))'$, similar to above:

$\boxed{f''(x) = e^x \cos(e^x) - (e^x)^2 \sin(e^x)}$

b. For the first derivative, we must use the chain rule

Let $F(x) = x^{10}$, $G(x) = 1 + 2x^2$. Then

$\boxed{h'(x) = 10(1+2x^2)^9 \cdot 4x}$

For the second derivative, we must use the product rule

Let $F(x) = 10(1+2x^2)^9$, $G(x) = 4x$. Then

$$h''(x) = (10(1+2x^2)^9)' \cdot 4x + (10(1+2x^2)^9) \cdot 4$$

To find $F'(x)$, we need to use the chain rule, with the outer function being $10x^9$ and the inner being $1+2x^2$ (similar to above) Then

$$h''(x) = (90(1+2x^2)^8) \cdot 4x \cdot 4x + 40(1+2x^2)^9$$

$$\boxed{= 1440x^2(1+2x^2)^8 + 40(1+2x^2)^9}$$

13. (20 points) Let $g(y) = \frac{y}{y-3}$. Remember you must justify your answers to receive credit.

a) What is the domain of $g(y)$?

Since division by 0 is not allowed,
the function $g(y)$ is not defined when $y-3=0$,
or equivalently when $y=3$.
Therefore the domain of g is $\{y \mid y \neq 3\}$,
or in interval notation,

$$(-\infty, 3) \cup (3, \infty)$$

b) Where is g continuous? Classify any discontinuities you find.

Since g is a rational function, we know
that it is continuous wherever it is defined.
Therefore, it is continuous on its domain, $(-\infty, 3) \cup (3, \infty)$.
The only discontinuity is at $y=3$.
Since we have

$$\lim_{y \to 3^+} \frac{y}{y-3} = \infty \quad + \quad \lim_{y \to 3^-} \frac{y}{y-3} = -\infty,$$

it has an infinite discontinuity at $y=3$.

(This problem is continued on the next page.)

c) What is the derivative of $g(y)$?

We apply the quotient rule;

$$g'(y) = \frac{(y)' \cdot (y-3) - y \cdot (y-3)'}{(y-3)^2}$$

$$= \frac{1 \cdot (y-3) - y \cdot 1}{(y-3)^2}$$

$$= -\frac{3}{(y-3)^2}$$

d) Find the equation of the tangent line to $z = g(y)$ at the point where $y = 4$ and $z = 4$.

m

The slope of the tangent line at $y=4$ is $g'(4)$.

Therefore $m = g'(4) = -\frac{3}{(4-3)^2} = -3.$

Using the point-slope form of the equation of a line, with the given point $(4,4)$, we obtain an equation of the tangent line as

$$z - 4 = -3(y-4).$$

* alternative solution: The slope-intercept form gives $z = -3y + b$. Substituting $y=4$ + $z=4$, we get $4 = -3 \cdot 4 + b$, or $b = 4 + 12 = 16$.

Therefore, the equation of the tangent line is

$$z = -3y + 16.$$

14. (15 points) Let $h(s) = \cos(s) + s^4 - 3s^2 + 5$. Remember you must justify your answers to receive credit.

a) State the definitions of even and odd for functions.

A function f is called even if $f(-x) = f(x)$ for every number x.

A function f is called odd if $f(-x) = -f(x)$ for every number x.

b) Find $h'(s)$.

Using the rule $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$, as well as the special derivative $\frac{d}{ds}[\cos(s)] = -\sin(s)$ and power rule, we have

$$h'(s) = \frac{d}{ds}[\cos(s)] + \frac{d}{ds}[s^4] - \frac{d}{ds}[3s^2] + \frac{d}{ds}[5]$$

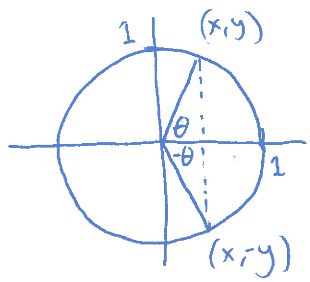
$$= -\sin(s) + 4s^3 - 3\frac{d}{ds}[s^2] + 0$$

$$= -\sin(s) + 4s^3 - 6s$$

(This problem is continued on the next page.)

c) Is $h(s)$ even, odd, both, or neither? Is $h'(s)$ even, odd, both, or neither?

First note that $\cos(\theta)$ is an even function. This is because the point (x,y) on the unit circle corresponding to the angle θ has the same x-value as the point corresponding to the angle $-\theta$:



$$\cos\theta = x$$

$$\cos(-\theta) = x$$

By similar reasoning, $\sin\theta$ is an odd function, i.e. $\sin(-\theta) = -\sin(\theta)$. So for any number s

$$\begin{aligned} h(-s) &= \cos(-s) + (-s)^4 - 3(-s)^2 + 5 \\ &= \cos(s) + s^4 - 3s^2 + 5 \\ &= h(s) \end{aligned}$$

and $h(s)$ is even. Similarly for any number s

$$\begin{aligned} h'(-s) &= -\sin(-s) + 4(-s)^3 - 6(-s) \\ &= -(-\sin(s)) - 4s^3 + 6s \\ &= -(-\sin(s) + 4s^3 - 6s) \\ &= -h'(s) \end{aligned}$$

and $h'(s)$ is odd.

15. (10 points) Let

$$f(x) = x^3 - 6x^2 - 15x + 128.$$

Find all the points on the graph $y = f(x)$ where the tangent line is horizontal. Remember you must justify your answers to receive credit.

+3 The tangent line to a graph is horizontal precisely when $f'(c)=0$.

+1

+4 Using the Power Rule, we compute $f'(x) = 3x^2 - 12x - 15$

And then by setting $f'(x)=0$, we solve for x, obtaining...

$$\begin{aligned} f'(x)=0 &\Rightarrow 3(x^2-4x-5)=0 \\ &\Rightarrow 3(x-5)(x+1)=0 \\ &\Rightarrow \boxed{x=5, \ x=-1} \end{aligned}$$

+1

To find the points on the graph, we then evaluate $f(5)$ and $f(-1)$ to find the y-values:

$$\begin{aligned} f(5) &= (5)^3 - 6(5)^2 - 15(5) + 128 \\ &= 125 - 150 - 75 + 128 \quad \Rightarrow \boxed{(5, 28)} \\ &= -100 + 128 = 28 \end{aligned}$$

+1

$$\begin{aligned} f(-1) &= (-1)^3 - 6(-1)^2 - 15(-1) + 128 \\ &= -1 - 6 + 15 + 128 \quad \Rightarrow \boxed{(-1, 136)} \\ &= 8 + 128 = 136 \end{aligned}$$

16. (10 points) Let $h(x)$ be defined by

$$h(x) = \begin{cases} x^2 \sin(\frac{x^3-4x+3}{x}) & x \neq 0\,, \\ c & x = 0\,, \end{cases}$$

where c is a constant. For which value of c is $h(x)$ continuous at $x = 0$? Hint: Recall that $-1 \leq \sin(z) \leq 1$ for all z. Remember you must justify your answers to receive credit.

$-1 \leq \sin\left(\frac{x^3-4x+3}{x}\right) \leq 1$ for all $x \neq 0$.

Multiplying by x^2 gives us

$-x^2 \leq x^2 \sin\left(\frac{x^3-4x+3}{x}\right) \leq x^2$, for all $x \neq 0$.

Taking limits as x approaches 0, we get

$\lim_{x\to 0} -x^2 \leq \lim_{x\to 0} x^2 \sin\left(\frac{x^3-4x+3}{x}\right) \leq \lim_{x\to 0} x^2$.

Since $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$, by the squeeze theorem, we get

$\lim_{x\to 0} x^2 \sin\left(\frac{x^3-4x+3}{x}\right) = 0$, so $\lim_{x\to 0} h(x) = 0$.

For $h(x)$ to be continuous at $x=0$, we need $\lim_{x\to 0} h(x) = h(0)$.

Since $\lim_{x\to 0} h(x) = 0$, and $h(0) = c$, setting $c = 0$ makes $h(x)$ continuous at $x=0$.