Expanding series using forward differences

Mark S Lovett

[1]Santa Fe College

Abstract

In this work we introduce a method to expand series as the sum of forward differences of the original summand. We found that a series can be transformed into a new series partly consisting of the forward difference also known as the discrete derivatives of the original summand of a series. This method can be used to calculate the sum of a series over an arbitrary finite set of natural numbers. It can be applied to various areas of calculus such as the calculation of Riemann sums, as well as possible applications as a convergence and divergence test for infinite series.

Introduction

The inspiration for my research came from looking at the relationship betweem terms of an integral. I realized, that each term could be written as the first term plus a series of the forward differences between terms up to the term that we were writing. This allowed me to expand a series and rewrite it into multiple new series. Since any term in a series can be written as its first term and the differences between the terms, I can use what is called a discrete derivative also known as a forward difference to rewrite the series. A forward difference is defined as

$$h \neq 0 \qquad \Delta_h^{c+1} f(z) = \frac{\Delta_h^c f(z+h) - \Delta_h^c f(z)}{h}.$$

Where h is the separation in domain values and c is a constant. For my purposes I will let $h = 1$ because the domain of a series consists of consecutive integers. But I will drop h and just write $\Delta^c f(z)$, because h won't change through out my research. The forward difference will allow me to find the difference between consecutive terms in a series and use these differences to derive a new series with a equivalent sum. The original summand rewritten as a forward difference will look like

$$f(z) = \Delta^0 f(z).$$

For my work with a series we can only take the forward difference $n-1$ times where n is the number of terms of the original series. Consider a series with only two terms, there can only exist one difference between the terms and this difference is obtained in the first forward difference. For any other magnitude forward differences it would not make sense since there would be no other differences to take the difference between.

SF Santa Fe College

- Email: mark.s.lovett@my.sfcollege.edu

Programmed in LaTeX

Results

In my research I observed that any series can be rewritten using the forward differences between its terms. I substituted in the initial term plus a series of the forward differences for each of the original terms of the original series. After I simplified I found that I was left with a new series that had an equivalent sum of the original series. I continued to do this process on the result and discovered another series. I then observed a pattern that a series could be rewritten as many as $n-1$ times using this method. Below, I have given some examples of the different representations of the sum that I found. equation (1) represents the first series I found after I substituted in the forward differences, equation (2) represents the second, and equation (3) represents the third substitution. Equation (4) represents the result when $n-1$ substitutions are made.

$$\sum_{z=q}^{p} f(z) = \Delta^0 f(q)n + \sum_{i=1}^{[n-1]} \sum_{z=q}^{[p-i]} \Delta^1 f(z) \tag{1}$$

$$\sum_{z=q}^{p} f(z) = \Delta^0 f(q)n + \Delta^1 f(q) \sum_{k=1}^{[n-1]} k + \sum_{j=1}^{[n-2]} \sum_{i=1}^{[(n-1)-j]} \sum_{z=q}^{[(p-i)-j]} \Delta^2 f(z) \tag{2}$$

$$\sum_{z=q}^{p} f(z) = \Delta^0 f(q)n + \Delta^1 f(q) \sum_{k=1}^{[n-1]} k + \Delta^2 f(q) \sum_{i=1}^{[n-2]} \sum_{k=1}^{[(n-1)-i]} k + \sum_{m=1}^{[n-3]} \sum_{j=1}^{[(n-2)-m]} \sum_{i=1}^{[(n-1)-j-m]} \sum_{z=q}^{[p-i-m-j]} \Delta^3 f(z) \tag{3}$$

$$\sum_{z=1}^{p} f(z) = \Delta^0 f(q)n + \Delta^1 f(q) \sum_{k=1}^{[n-1]} k + \Delta^2 f(q) \sum_{i=1}^{[n-2]} \sum_{k=1}^{[(n-1)-i]} k + \Delta^3 f(q) \sum_{j=1}^{[n-3]} \sum_{i=1}^{[(n-2)-j]} \sum_{k=1}^{[(n-1)-i-j]} k + ... + \Delta^{n-1} f(q) \tag{4}$$

After I found equation (4) I was then able to apply the summation formulas for an arithmetic series and the various sum of a polynomial functions to each of the sums in equation (4). The result is equation (5) a simplified version of equation (4).

$$\sum_{z=q}^{p} f(z) = \sum_{k=1}^{n} \Delta^{k-1} f(q) \frac{n!}{k!(n-k)!} = \sum_{k=1}^{n} \Delta^{k-1} f(q) \binom{n}{k} \tag{5}$$

Examples

When we apply equation (5) to $\Sigma_{z=1}^{p} 2^z$ and $\Sigma_{z=1}^{p} \left(\frac{1}{2}\right)^z$, we get the following equations respectively. For this example $k = q$ and $n = p$ this is a special case, in other cases they can vary.

$$\sum_{z=1}^{p} 2^z = \sum_{k=1}^{n} 2\binom{n}{k}$$

$$\sum_{z=1}^{p} \left(\frac{1}{2}\right)^z = \sum_{k=1}^{n} (-1)^{k-1} * \left(\frac{1}{2}\right)^k \binom{n}{k}$$

Below is a table comparing the terms of each series when $n = 5$ as well as the sum for each of the series.

Table of terms when $n = 5$

k	2^k	$2\binom{5}{k}$	$\left(\frac{1}{2}\right)^k$	$\frac{1}{2}\left(\frac{-1}{2}\right)^{k-1}\binom{5}{k}$
1	2	10	1/2	2.5
2	4	20	1/4	-2.5
3	8	20	1/8	1.250
4	16	10	1/16	-.3125
5	32	2	1/32	.03125
Sum	62	62	0.96875	0.96875

Graph of the partial sums

If we graph the partial sums of the equations, then they exactly fit the graphs of the partial sums of the original series. Therefore if we use equation (5) to rewrite a series the result shares the originals series convergence or divergence. It is important to note that both series only differ in the forward differences of their summand evaluated at their lower bound.

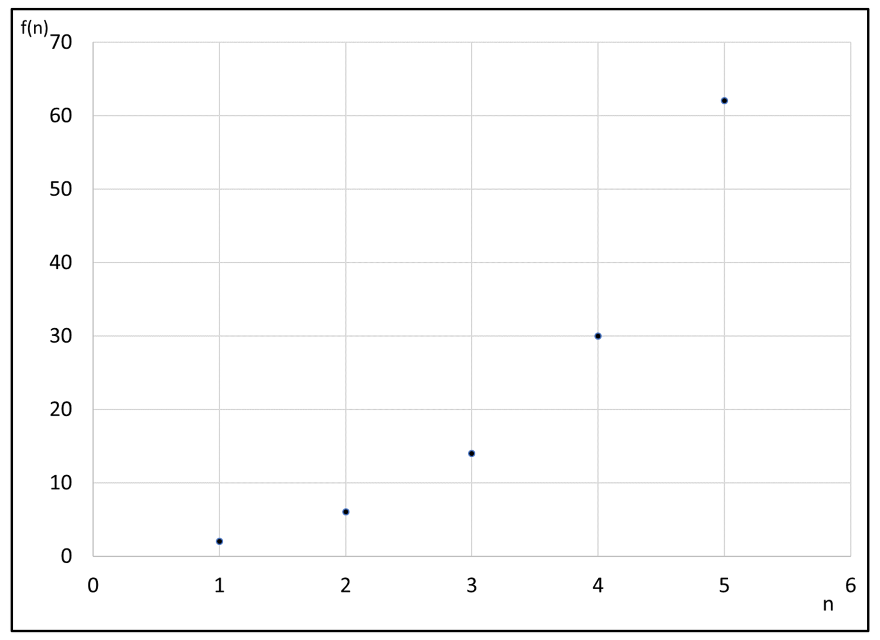

Figure: $f(n) = \Sigma_{n=1}^{k} 2\binom{n}{k}$

Graph of the partial sums

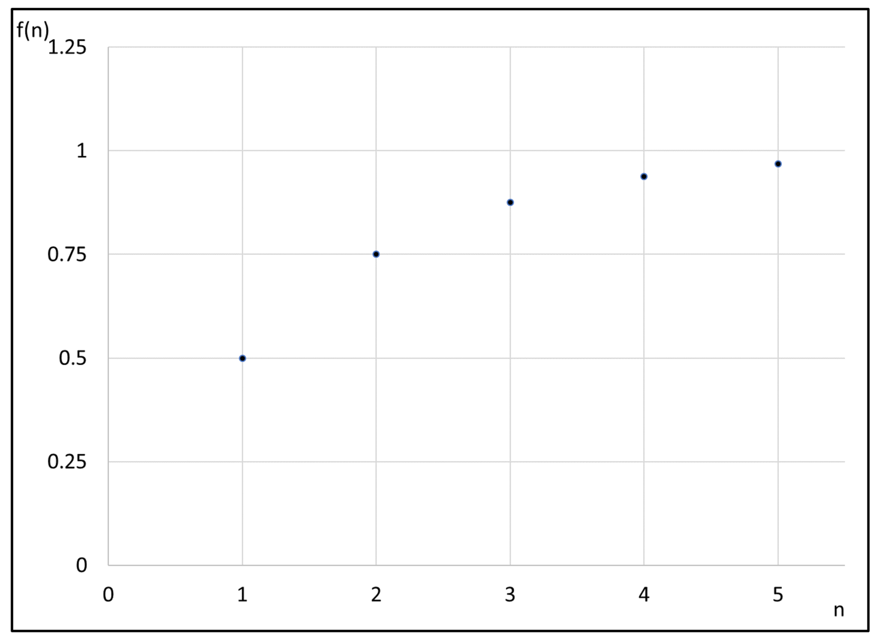

Figure: $f(n) = \Sigma_{n=1}^{k} \frac{1}{2}\left(\frac{-1}{2}\right)^{k-1}\binom{n}{k}$

Conclusion

Altogether, I found that a series can be expanded into as many as $n-1$ different series and that the last series can be simplified into just the sum of the combinations of n and k times the forward differences evaluated at the lower bound of the original series. These series can be used to possibly test the convergence or divergence in a series, because the only difference between a convergent and divergent is the forward differences evaluated at the lower bound of an original series. The resulting series can also be used to calculate the sum of any series with an arbitrary finite set of natural numbers as its domain. It may even be possible to use any of the resulting series to rewrite a Riemann sum and possibly create a new way to integrate.

References

[1] David Gleich.
A tutorial for solving nasty sums.
https://www.cs.purdue.edu/homes/dgleich/publications/Gleich%202005%20-%20finite%20calculus.pdf,
2005.

Acknowledgments

I would like to thank Dr. Adam Christopherson, Brandy Cooke, Aharon Dagan, Dr. Karim Diff, Mark McGraw, Jason Rummler, Dr. Eric Stetler, and Dr. Lilia Tache for their help in my research. I also would like to thank Santa Fe College and Elizabeth Hays Travel Award for funding my trip.