Math essentials

Vector operators

Vectors – an object with a magnitude and a direction (*scalar has only a magnitude)

$$\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k} = (A_1, A_2, A_3)$$

A_1, A_2, A_3: component of $\vec{A}$
$\hat{\imath}, \hat{\jmath}, \hat{k}$: coordinate unit vectors
*Magnitude of a vector

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

Inner product

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta\,, 0 \le \theta \le \pi$$

$$\vec{A} \cdot \vec{B} = \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right) \cdot \left(B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}\right) = A_1B_1 + A_2B_2 + A_3B_3$$

Cross product

$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}|\,\vec{u}\sin\theta$$

where $\vec{u} \perp \vec{A}\,\&\,\vec{B}$ $(right\ hand\ rule)$, $|\vec{u}| = 1$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \hat{\imath}(A_2B_3 - A_3B_2) - \hat{\jmath}(A_1B_3 - A_3B_1) + \hat{k}(A_1B_2 - A_1B_2)$$

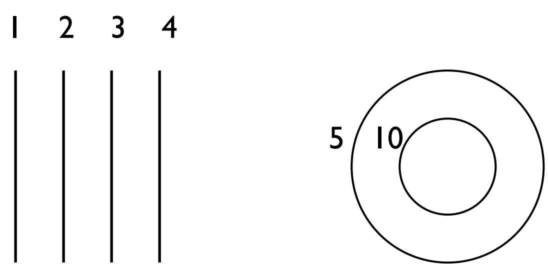
Gradient operator (operator on a scalar field)

$$\nabla \equiv \hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{k}\frac{\partial}{\partial \mathrm{z}}$$

$$\nabla\emptyset = \hat{\imath}\frac{\partial\emptyset}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial\emptyset}{\partial \mathrm{y}} + \hat{k}\frac{\partial\emptyset}{\partial \mathrm{z}}$$

= a <u>vector</u> points toward large values of the scalar

Ex)

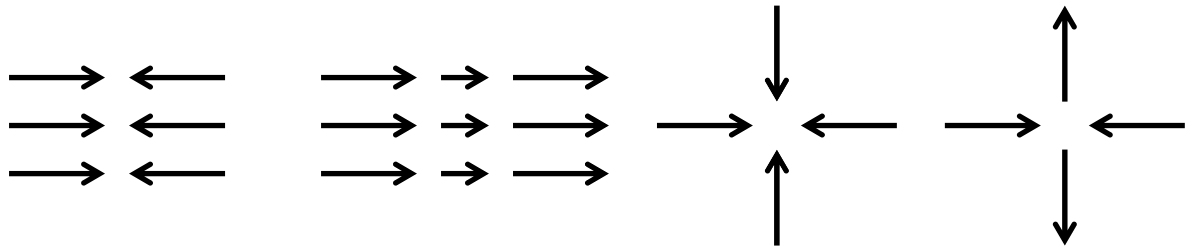


Divergence (operator on a vector field)

$$\nabla \cdot \vec{A} = \left(\hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{k}\frac{\partial}{\partial \mathrm{z}}\right) \cdot \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right) = \frac{\partial A_1}{\partial \mathrm{x}} + \frac{\partial A_2}{\partial \mathrm{y}} + \frac{\partial A_3}{\partial \mathrm{z}}$$

=

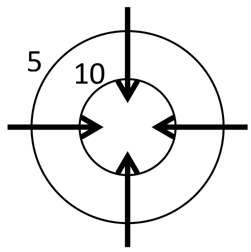
Ex)



Laplacian (operator on a scalar field, divergence of gradient)

$$\nabla \cdot \vec{A} = \nabla \cdot (\nabla \emptyset) = \nabla^2 \emptyset = \frac{\partial^2 \emptyset}{\partial \mathrm{x}^2} + \frac{\partial^2 \emptyset}{\partial \mathrm{y}^2} + \frac{\partial^2 \emptyset}{\partial \mathrm{z}^2}$$

Ex) In this example, we have convergence (negative divergence) at the peak ($\nabla^2 \emptyset \propto -\emptyset$)



Curl (operator on a vector field)

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial \mathrm{x}} & \dfrac{\partial}{\partial \mathrm{y}} & \dfrac{\partial}{\partial \mathrm{z}} \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{\imath}\left(\frac{\partial A_3}{\partial \mathrm{y}} - \frac{\partial A_2}{\partial \mathrm{z}}\right) - \hat{\jmath}\left(\frac{\partial A_3}{\partial \mathrm{x}} - \frac{\partial A_1}{\partial \mathrm{z}}\right) + \hat{k}\left(\frac{\partial A_2}{\partial \mathrm{x}} - \frac{\partial A_1}{\partial \mathrm{y}}\right)$$

Divergence theorem

$$\int_V (\nabla \cdot \vec{A})\, dV = \int_S (\vec{A} \cdot \hat{n})\, dS$$

where V: volume, S: surface sounding V, $\hat{n}$: unit normal vector outward from S

Stokes theorem

$$\int_S \{(\nabla \times \vec{A}) \cdot \hat{n}\}\, dS = \oint_C (\vec{A} \cdot \hat{r})\, dC$$

where C: curve bounding S $\hat{r}$: “follows” the RH rule counter-clockwise

Derivatives

“**Ordinary derivative**”, $\frac{\mathrm{d}}{\mathrm{dt}}$

$f(t)$ is a function of one independent variable, t, and $\frac{\mathrm{df}}{\mathrm{dt}}$ is the rate of change of f with respect to t. $\frac{\mathrm{df}}{\mathrm{dt}}$ may also be a function of t, or a constant.

Ex)

$$\mathrm{f(x)} = \sin x \;\rightarrow\; \frac{\mathrm{df}}{\mathrm{dx}} = -\cos x$$

"**Partial derivative**", $\frac{\partial}{\partial \mathrm{t}}$

$f(x, y, t)$ is a function of several independent variables, (x,y,t), and $\frac{\partial \mathrm{f}}{\partial \mathrm{t}}$ is the rate of change of f with respect to t <u>when the other independent variables are held constant</u>. $\frac{\partial \mathrm{f}}{\partial \mathrm{t}}$ may also be a function of the independent variables.

Ex)

$$\mathrm{f(x, y, t)} = 3\mathrm{x}^2 y \sin t \;\rightarrow\; \frac{\partial \mathrm{f}}{\partial \mathrm{t}} = -3\mathrm{x}^2 y \cos t$$

The order of partial derivatives is commutative if the function is smooth

$$\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}\, \partial \mathrm{y}} = \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}\, \partial \mathrm{x}}$$

Chain rule:

$$f(u(x), t) \;\rightarrow\; \frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \frac{\partial \mathrm{f}}{\partial \mathrm{u}} \frac{\partial \mathrm{u}}{\partial \mathrm{x}}$$

Taylor series – represents a function by an infinite sum of terms involving derivative evaluated at a point (!: factorial)

$$f(x) = f(a) + \left.\frac{\mathrm{df}}{\mathrm{dx}}\right|_{x=a} (x-a) + \left.\frac{1}{2}\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{dx}^2}\right|_{x=a} (x-a)^2 + \left.\frac{1}{3!}\frac{\mathrm{d}^3\mathrm{f}}{\mathrm{dx}^3}\right|_{x=a} (x-a)^3 + \cdots$$

$$f(x, y) = f(a, b) + \left.\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\right|_{a,b} (x-a) + \left.\frac{\partial \mathrm{f}}{\partial \mathrm{y}}\right|_{a,b} (y-b) + \frac{1}{2}\left[\left.\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2}\right|_{a,b} (x-a)^2 + \left.\frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}^2}\right|_{a,b} (y-b)^2 + 2\left.\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}\, \partial \mathrm{y}}\right|_{a,b} (x-a)(y-b)\right] + \cdots$$

"**material derivative**", "**total derivative**", $\frac{\mathrm{D}}{\mathrm{Dt}}$ – meaning the changes following the movement of the air parcel.

$$T(x, y, z, t) = T(a, b, c, d) + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{a,b,c,d} (x-a) + + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right|_{a,b,c,d} (y-b) + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{z}}\right|_{a,b,c,d} (z-c) + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{t}}\right|_{a,b,c,d} (t-d) + \cdots$$

Substitute

$$x = x + \delta x, y = y + \delta y, z = z + \delta z, t = t + \delta t$$
$$a = x, b = y, c = z, d = t$$

Then,

$$T(x + \delta x, y + \delta y, z + \delta z, t + \delta t) - T(x, y, z, t)$$
$$= \left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{x,y,z,t} (x + \delta x - x) + + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right|_{x,y,z,t} (\delta y) + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{z}}\right|_{x,y,z,t} (\delta z) + \left.\frac{\partial \mathrm{T}}{\partial \mathrm{t}}\right|_{x,y,z,t} (\delta t) + \cdots$$

Define

$$\delta \mathrm{T} \equiv T(x + \delta x, y + \delta y, z + \delta z, t + \delta t) - T(x, y, z, t)$$

$$\delta \mathrm{T} = \frac{\partial \mathrm{T}}{\partial \mathrm{t}} \delta \mathrm{t} + \frac{\partial \mathrm{T}}{\partial \mathrm{x}} \delta \mathrm{x} + \frac{\partial \mathrm{T}}{\partial \mathrm{y}} \delta \mathrm{y} + \frac{\partial \mathrm{T}}{\partial \mathrm{z}} \delta \mathrm{z}$$

$$\frac{\delta \mathrm{T}}{\delta \mathrm{t}} = \frac{\partial \mathrm{T}}{\partial \mathrm{t}} + \frac{\partial \mathrm{T}}{\partial \mathrm{x}} \frac{\delta \mathrm{x}}{\delta \mathrm{t}} + \frac{\partial \mathrm{T}}{\partial \mathrm{y}} \frac{\delta \mathrm{y}}{\delta \mathrm{t}} + \frac{\partial \mathrm{T}}{\partial \mathrm{z}} \frac{\delta \mathrm{z}}{\delta \mathrm{t}}$$

$$\delta \mathrm{t}, \delta \mathrm{x}, \delta \mathrm{y}, \delta \mathrm{z} \ \rightarrow 0$$

Then,

$$\frac{\delta \mathrm{x}}{\delta \mathrm{t}} = u, \frac{\delta \mathrm{y}}{\delta \mathrm{t}} = v, \frac{\delta \mathrm{z}}{\delta \mathrm{t}} = w$$

$$\frac{\mathrm{DT}}{\mathrm{Dt}} = \frac{\partial \mathrm{T}}{\partial \mathrm{t}} + \mathrm{u} \frac{\partial \mathrm{T}}{\partial \mathrm{x}} + \mathrm{v} \frac{\partial \mathrm{T}}{\partial \mathrm{y}} + \mathrm{w} \frac{\partial \mathrm{T}}{\partial \mathrm{z}} = \frac{\partial \mathrm{T}}{\partial \mathrm{t}} + \vec{\mathrm{U}} \cdot \nabla T$$