Sample Solutions for Assignment 6.

Reading: Lectures 16 and 20-21 in the text.

1. Let $A$ be an $m$ by $m$ nonsingular matrix and let $b$ be a given nonzero $m$-vector. Suppose $x$ satisfies $Ax = b$ and $\hat{x}$ satisfies $A\hat{x} = \hat{b}$, where $\hat{b}$ is slightly different from $b$. Show that

$$\frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq \kappa(A) \frac{\|\hat{b} - b\|_2}{\|b\|_2},$$

where $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$ is the 2-norm condition number of $A$. Show that there are nonzero vectors $b$ and $\hat{b} - b$ for which equality holds. [Note: You might want to review material in Lecture 12 of the text.]

Subtracting the equations for $x$ and $\hat{x}$, we find that $A(\hat{x} - x) = \hat{b} - b$, or $\hat{x} - x = A^{-1}(\hat{b} - b)$. Taking the 2-norm on each side (and letting $\| \cdot \|$ denote the 2-norm), we can write

$$\|\hat{x} - x\| \leq \|A^{-1}\| \cdot \|\hat{b} - b\|,$$

where equality will be obtained if $\hat{b} - b$ lies in the direction of the right singular vector of $A^{-1}$ corresponding to the largest singular value; that is, if $\hat{b} - b$ lies in the direction of the left singular vector of $A$ corresponding to the smallest singular value of $A$. Dividing each side by $\|x\|$, inequality (1) can be written in the form

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq \|A^{-1}\| \frac{\|\hat{b} - b\|}{\|b\|} \cdot \frac{\|b\|}{\|x\|}.$$  \hspace{1cm} (2)

Since $b = Ax$, the last factor in (2) satisfies $\|b\|/\|x\| \leq \|A\|$, with equality if $x$ lies in the direction of the right singular vector of $A$ corresponding to the largest singular value; that is, if $b = Ax$ is in the direction of the left singular vector of $A$ corresponding to the largest singular value. Thus we have

$$\frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq \kappa(A) \frac{\|\hat{b} - b\|_2}{\|b\|_2},$$

with equality if $b$ is a nonzero vector in the direction of the right singular vector of $A$ corresponding to the largest singular value and $\hat{b} - b$ is a nonzero vector in the direction of the right singular vector of $A$ corresponding to the smallest singular value.

2. The idea of this exercise is to carry out an experiment analogous to the one described in Lec. 16 of the text, but for the SVD instead of QR factorization.
(a) Write a program that constructs a 50 × 50 matrix $A = U*S*V'$, where $U$ and $V$ are random orthogonal matrices and $S$ is a diagonal matrix whose diagonal entries are uniformly distributed numbers in $[0,1]$, sorted into nonincreasing order. You can use the following lines in MATLAB:

$$[U,X] = \text{qr}(\text{randn}(50));$$
$$[V,X] = \text{qr}(\text{randn}(50));$$
$$S = \text{diag}((\text{sort}(\text{rand}(50,1),'\text{descend}')));$$
$$A = U*S*V';$$

Compute the SVD of $A$: $[U2,S2,V2] = \text{svd}(A)$; Recall that the SVD of a real square matrix is not quite uniquely determined. Make sure that the signs of the columns of $U2$ and $V2$ match those of $U$ and $V$ as follows:

for $j=1:50$,
    if $U2(:,j)'*U(:,j) < 0$, % The signs are different for the jth column.
        $U2(:,j) = -U2(:,j);$ \text{i.e., change the sign to match the sign of U.}$
        $V2(:,j) = -V2(:,j);$ \text{i.e., change the sign to match the sign of V.}$
    end;
end;

Now compute $\text{norm}(U2-U)$, $\text{norm}(V2-V)$, $\text{norm}(S2-S)/\text{norm}(S)$, and $\text{norm}(A - U2*S2*V2')/\text{norm}(A)$. Run your program with five different random matrices and comment on whether the various differences seem to be connected with the condition number of $A$, cond($A$).

(b) For each of the matrices in part(a), replace the diagonal entries in $S$ by their sixth powers (thus making the condition number of $A$ much larger) and repeat the experiment. Do you see significant differences between these results and those of the experiment for QR factorization in the text?

I saw no clear dependence on condition number in part (a). In part (b), it appeared that the differences between $U$ and $U2$ (labeled diffU) and between $V$ and $V2$ (labeled diffV) were somewhat dependent on condition number, but the relative differences in singular values (labeled diffS) and the difference $\text{norm}(A - U2*S2*V2')/\text{norm}(A)$ (labeled diffA) appeared independent of condition number. Results are shown below, with test matrices ordered by increasing condition number.

<table>
<thead>
<tr>
<th>cond. no</th>
<th>diffU</th>
<th>diffV</th>
<th>diffS</th>
<th>diffA</th>
</tr>
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<td>1.1769e-12</td>
<td>1.1765e-12</td>
<td>1.4020e-15</td>
<td>6.1231e-15</td>
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<td>4.0391e-10</td>
<td>1.3680e-15</td>
<td>6.7555e-15</td>
</tr>
</tbody>
</table>
3. Write the following matrix in the form $LU$, where $L$ is a unit lower triangular matrix and $U$ is an upper triangular matrix:

$$
\begin{bmatrix}
4 & -1 & -1 \\
-1 & 4 & -1 \\
-1 & -1 & 4
\end{bmatrix}.
$$

Write the same matrix in the form $LL^T$, where $L$ is lower triangular.

Let $L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $L_1A = \begin{pmatrix} 4 & -1 & -1 \\ 0 & \frac{\sqrt{15}}{4} & -\frac{5}{4} \\ 0 & -\frac{5}{4} & \frac{15}{4} \end{pmatrix}$.

Let $L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$.

Then $L_2L_1A = \begin{pmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{pmatrix}$.

Thus $A = LU$ where

$$L = (L_2L_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{pmatrix}.$$

If $D$ is the diagonal of $U$, then we can write $LU = (LD^{1/2})(D^{-1/2}U) \equiv \hat{L}\hat{U}$ where $\hat{L}$ and $\hat{U}$ have the same diagonal entries. That is

$$\hat{L} = L \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{\sqrt{15}}{2} & 0 \\ 0 & 0 & \sqrt{\frac{10}{3}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{15}}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{15}}{6} & \sqrt{\frac{10}{3}} \end{pmatrix},$$

$$\hat{U} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2\sqrt{15}}{3} & 0 \\ 0 & 0 & \sqrt{\frac{5}{10}} \end{pmatrix} U = \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{15}}{2} & -\frac{\sqrt{15}}{6} \\ 0 & 0 & \sqrt{\frac{10}{3}} \end{pmatrix}.$$  

Note that $\hat{U} = \hat{L}^T$, so this is the Cholesky factorization.

Since the row operations of Gaussian elimination leave the determinants \( \det(A_{1,k,1:k}) \) unchanged, if these determinants are all nonzero, then as we reduce \( A \) to upper triangular form, the diagonal entries of the upper triangular blocks must be nonzero (since the determinant of an upper triangular matrix is the product of its diagonal entries):

\[
\begin{bmatrix}
  u_{11} & u_{12} & \ldots & u_{1,k-1} & u_{1k} & \ldots & u_{1m} \\
  u_{22} & \ddots & & & \ddots & & \\
  \vdots & & \ddots & & & & \\
  u_{k-1,k-1} & u_{k-1,k} & \ldots & u_{k-1,m} & u_{kk} & \ldots & u_{km} \\
  \vdots & & & & & \ddots & \\
  u_{mk} & & & & u_{mm} & & \\
\end{bmatrix},
\]

Thus, we always have a nonzero pivot element \( u_{kk} \). Conversely, if a block is singular, say, \( A_{1,k,1:k} \) is the first singular block, then \( u_{11}, \ldots, u_{k-1,k-1} \) are nonzero but \( u_{kk} = 0 \) since the determinant of the \( k \) by \( k \) upper left block of \( U \) must be zero. In this case we must pivot, and the matrix does not have an \( LU \) decomposition.

Suppose \( A = L_1U_1 = L_2U_2 \) where \( L_1 \) and \( L_2 \) are unit lower triangular matrices and \( U_1 \) and \( U_2 \) are upper triangular matrices. Then \( L_2^{-1}L_1 = U_2U_1^{-1} \), and since the matrix on the left is lower triangular and the one on the right is upper triangular, each matrix must be diagonal. The matrix on the left has 1’s on its diagonal, so it must just be the identity, which implies that \( L_2 = L_1 \). And since the matrix on the right is equal to this, it is also the identity and hence \( U_2 = U_1 \). Thus, the \( LU \) factorization is unique.

5. p. 154, Exercise 20.2.

\( L \) and \( U \) are banded, with half-bandwidth \( p \). This is illustrated below:

\[
\begin{bmatrix}
  x & x & x & 0 & \ldots & 0 \\
  x & x & x & \ddots & \vdots & \\
  x & x & x & \ddots & 0 & \\
  0 & x & x & \ddots & x & \\
  \vdots & \ddots & \ddots & \ddots & x & \\
  0 & \ldots & 0 & x & x & x \\
\end{bmatrix} \rightarrow \begin{bmatrix}
  x & x & x & 0 & \ldots & 0 \\
  0 & x & x & \ddots & \vdots & \\
  0 & x & x & \ddots & 0 & \\
  0 & x & x & \ddots & x & \\
  \vdots & \ddots & \ddots & \ddots & x & \\
  0 & \ldots & 0 & x & x & x \\
\end{bmatrix} \rightarrow
\]
Since we need only eliminate in \( p \) rows below the diagonal, \( L_{ij} \) will be 0 for \( i - j > p \), and since the elimination steps do not introduce any new nonzeros outside the band, \( U_{ij} \) will be 0 for \( j - i > p \).

At stage \( j \) of Gaussian elimination, we must eliminate the entry in column \( j \) from rows \( j + 1 \) through \( \min\{j + p, m\} \), since the later rows have zeros in column \( j \) anyway. Only entries in columns \( j \) through \( \min\{j + p, m\} \) are affected because the later entries in row \( j \) are 0. Thus, the work at stage \( j \) is about \( 2 \min\{p, m - j\} \cdot \min\{p + 1, m - j + 1\} \), and the total work is about

\[
\sum_{j=1}^{m-1} 2 \min\{p, m - j\} \cdot \min\{p + 1, m - j + 1\} \sim 2p^2m.
\]


If \( A \) is banded with half bandwidth \( p \) and we use partial pivoting, then the half bandwidth of \( U \) can be as large as \( 2p \). The band structure of \( L \) may be destroyed, but it will still have at most \( p \) nonzero entries below the diagonal in each column. To see this, note that the first pivot entry will be chosen from among the first \( p + 1 \) rows. In the worst case, this will mean interchanging rows 1 and \( p + 1 \), thus moving row \( p + 1 \) to the top. The first row now may have nonzeros in in positions 1 through \( 2p + 1 \). The first elimination step is performed only on the \( p \) rows below the first which have nonzeros in column 1. Entries in columns 2 through \( 2p + 1 \) of these rows may be nonzero after the elimination. This is illustrated below for a pentadiagonal matrix, where \( p = 2 \):
After the first step, there are still $p$ rows below row 2 from which the pivot can be chosen. In the worst case, row $p+2$ is moved into the pivot position and then when the elimination is performed, entries out to column $2p+2$ fill in with nonzeros in the rows that had nonzeros in column 2, as illustrated below:

$$
\begin{bmatrix}
  x & x & x & x & 0 & 0 \\
  0 & x & x & x & x & 0 \\
  0 & x & x & x & x & 0 \\
  0 & x & x & x & x & x \\
  0 & 0 & 0 & x & x & x \\
  0 & 0 & 0 & 0 & x & x \\
\end{bmatrix}.
$$

Elimination continues in this way, resulting in an upper triangular factor $U$ with half bandwidth at most $2p$. Due to pivoting, the band structure of $L$ may be destroyed, but since, at each step, elimination is performed in only $p$ rows, $L$ still has at most $p$ nonzeros per column below the main diagonal.