Sample Solutions for Assignment 4.

Reading: Lectures 9-11 in the text.


(a) Since $A$ is upper triangular, its eigenvalues are its diagonal entries, which are all 1’s. The determinant of $A$ is the product of the eigenvalues; i.e., $1$. The rank of $A$ is $m$ since it is an $m$ by $m$ upper triangular matrix whose diagonal entries are all nonzero.

(b) Computing $A^{-1}$ column by column, we find

$$A^{-1} = \begin{bmatrix}
1 & -2 & 4 & -8 & \ldots & (-2)^{m-1} \\
0 & 1 & -2 & 4 & \ldots & (-2)^{m-2} \\
0 & 0 & 1 & -2 & \ldots & (-2)^{m-3} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -2 \\
0 & 0 & 0 & \ldots & \ldots & 1
\end{bmatrix}.$$  

(c) The smallest singular value $\sigma_m$ of $A$ is $1/\|A^{-1}\|_2$. If $A^{-1}$ is applied to the last unit vector then that picks out the last column of $A^{-1}$, which has 2-norm

$$\left(\sum_{j=0}^{m-1} 4^j \right)^{1/2} \geq \left[\frac{4^m - 1}{3}\right]^{1/2} \geq 2^{m-1}.$$  

Thus $1/\sigma_m \geq 2^{m-1}$, or, $\sigma_m \leq 1/2^{m-1}$. For $m > 1$, $\sigma_m$ is much smaller than any eigenvalue of $A$.

2. p. 76, Exercise 10.1

A Householder reflector has the form

$$I - 2\frac{vv^*}{v^*v},$$

for some nonzero vector $v \in \mathbb{C}^m$.

(a) The vector $v$ is mapped to $-v$, so this is an eigenvector corresponding to eigenvalue $-1$. Any vector in the $(m-1)$-dimensional space $H$ orthogonal to $v$ is mapped to itself, so the other eigenvalues are all 1. Thus vectors in the hyperplane $H$ through which we are reflecting are mapped to themselves, while vectors orthogonal to this hyperplane (i.e., multiples of $v$) are mapped to their negatives.
(b) The determinant is the product of the eigenvalues, hence $-1$.

(c) Since this is a Hermitian matrix, the singular values are the absolute values of the eigenvalues; that is, they are all 1.

3. p. 76, Exercise 10.4 (a) and (b).

(a) Left multiplication by $J$ takes the vector $(1,0)^T$, for example, to the vector $(\cos \theta, -\sin \theta)^T$; thus it rotates this vector and all vectors in $\mathbb{R}^2$ clockwise through the angle $\theta$.

Left multiplication by $F$ maps the vector $(s,c+1)^T$ to itself, and it maps the vector $(s,c-1)^T$, which is orthogonal to $(s,c+1)^T$, to $-(s,c-1)^T$. Thus it reflects through the line that goes through the origin and the point $(s,c+1)$. [If $\theta = \pi$, so that $s = c + 1 = 0$, then it reflects through the x-axis.]

(b) Let $J_1$ have the form

$$
\begin{bmatrix}
  c & s \\
  -s & c \\
\end{bmatrix}
$$

where $s$ and $c$ are chosen to annihilate the $(2,1)$-entry of $A$; i.e.,

$$
-sa_{11} + ca_{21} = 0 \Rightarrow c = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}}, \quad s = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}}.
$$

Apply $J_1$ to $A$ to obtain a matrix of the form

$$
\begin{bmatrix}
  \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \ldots & \tilde{a}_{1m} \\
  0 & \tilde{a}_{22} & \tilde{a}_{23} & \ldots & \tilde{a}_{2m} \\
  a_{31} & a_{32} & a_{33} & \ldots & a_{3m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mm}
\end{bmatrix}
$$

Next take a matrix $J_2$ of the form

$$
\begin{bmatrix}
  c & 0 & s \\
  0 & 1 & 0 \\
  -s & 0 & c \\
\end{bmatrix}
$$

where $c$ and $s$ are chosen to annihilate the $(3,1)$-entry of $A$. Note that left multiplication by $J_2$ affects only rows 1 and 3, so the 0 that was previously created in the $(2,1)$-position remains 0. We continue until all entries below the diagonal in column 1 have been annihilated:

$$
\begin{bmatrix}
  \tilde{a}_{11} & \tilde{a}_{12} & \ldots & \tilde{a}_{1m} \\
  0 & \tilde{a}_{22} & \ldots & \tilde{a}_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \tilde{a}_{m2} & \ldots & \tilde{a}_{mm}
\end{bmatrix}
$$
We then apply matrices with rotations corresponding to rows 2 and \( j \), \( j > 2 \), and identities elsewhere, until the entries below the diagonal in column 2 are eliminated. These do not affect the zeros in column 1. We continue in this way through all of the columns until we obtain an upper triangular matrix.

4. \( p. 85 \), Exercise 11.3. [You may skip the methods in (b) and (c).].

Following is the Matlab code that I used and the resulting output:

```matlab
m = 50; n = 12; % 50 points, poly of degree 11
t = linspace(0,1,50); % equally spaced points between 0 and 1
A = vander(t); A = fliplr(A); % Vandermonde matrix
A = A(:,1:n); % Take only the first n columns
b = cos(4*t'); % Right-hand side vector

x_normal = (A'*A)
A'*b); % Solve using normal equations.
[Q,R] = qr(A,0); % Reduced QR decomposition.
x_qr = R
Q'*b);

x_backslash = A\b; % Solve using backslash.

[U,Sigma,V] = svd(A,0); % Reduced SVD.
Vpx = Sigma
U'*b);
x_svd = V*Vpx;

format long e % Print results.
x_normal, x_qr
[x_backslash, x_svd]
```

<table>
<thead>
<tr>
<th>x_normal</th>
<th>x_qr</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000000021274513e+00</td>
<td>1.0000000000996608e+00</td>
</tr>
<tr>
<td>-6.426274880823923e-06</td>
<td>-4.227430053544422e-07</td>
</tr>
<tr>
<td>-7.99975536351574e+00</td>
<td>-7.99981235686737e+00</td>
</tr>
<tr>
<td>-3.650928546790763e-03</td>
<td>-3.187632180147387e-04</td>
</tr>
<tr>
<td>1.069496442364852e+01</td>
<td>1.066943079579286e+01</td>
</tr>
<tr>
<td>-1.291352808941941e-01</td>
<td>-1.382028749663541e-02</td>
</tr>
<tr>
<td>-5.319925736917411e+00</td>
<td>-5.647075628816084e+00</td>
</tr>
<tr>
<td>-6.747547869339512e-01</td>
<td>-7.531602151596609e-02</td>
</tr>
<tr>
<td>2.402190965843583e+00</td>
<td>1.693606960044832e+00</td>
</tr>
<tr>
<td>-5.157659328522163e-01</td>
<td>6.032111365161300e-03</td>
</tr>
</tbody>
</table>
For the normal equations, the last eight entries of the first coefficient appear to be wrong, and all entries of the next coefficient (which is much smaller) appear to be wrong. The other methods seem to get the first coefficient right to all 16 places and the second one right to about 6 places. If one types \( \text{cond}(A) \) in Matlab, one finds that the condition number of \( A \) is about \( 10^8 \), which means that the condition number of \( A^*A \) is about \( 10^{16} \). It is actually surprising that one gets any correct digits by solving the normal equations. Some of the other coefficients, that are not particularly small, like the fourth one from the bottom, are completely wrong with the normal equations. I would expect errors of size about \( 10^{-8} \) with the other methods, and this seems to be the case for some of the coefficients, like the one that is fourth from the bottom.

5. Consider the following least squares approach for ranking sports teams. Suppose we have four college football teams, called simply T1, T2, T3, and T4. These four teams play each other with the following outcomes:

- T1 beats T2 by 4 points: 21 to 17.
- T3 beats T1 by 9 points: 27 to 18.
- T1 beats T4 by 6 points: 16 to 10.
- T3 beats T4 by 3 points: 10 to 7.
- T2 beats T4 by 7 points: 17 to 10.

To determine ranking points \( r_1, \ldots, r_4 \) for each team, we do a least squares fit to the overdetermined linear system:

\[
 r_1 - r_2 = 4,
\]
This system does not have a unique least squares solution, however.

(a) Show that if \((r_1, \ldots, r_4)^T\) solves the least squares problem above then so does the vector \((r_1 + c, \ldots, r_4 + c)\) for any constant \(c\).

Whatever the residual norm is with \(r_1, \ldots, r_4\), it is the same with \(r_1 + c, \ldots, r_4 + c\), since 
\[
4 - (r_1 - r_2) = 4 - ((r_1 + c) - (r_2 + c)), \ldots, 7 - (r_2 - r_4) = 7 - ((r_2 + c) - (r_4 + c)).
\]
Thus, if \(r_1, \ldots, r_4\) minimize the sum of squares of these differences, so do \(r_1 + c, \ldots, r_4 + c\).

To make the solution unique, we can fix the total number of ranking points, say, at 20. To do this, we add the following equation to those listed above:

\[
r_1 + r_2 + r_3 + r_4 = 20.
\]

(b) Explain why the least squares solution to the six equations listed will satisfy this last equation exactly.

Since the residuals in the first five equations are the same for \(r_1, \ldots, r_4\) and for \(r_1 + c, \ldots, r_4 + c\), once we have found some solution to the least squares problem for the first five equations, we can then choose \(c\) so that the sixth equation is satisfied exactly: 
\[
r_1 + r_2 + r_3 + r_4 + 4c = 20 \implies c = (20 - r_1 - r_2 - r_3 - r_4)/4.
\]
This must be the least squares solution to the set of six equations, since the sum of squares of residuals for the first five equations cannot be made any smaller and the residual in the sixth equation is zero.

(c) Use Matlab to determine the values \(r_1, r_2, r_3, r_4\) that most closely satisfy these equations, and based on your results, rank the four teams. [This method of ranking sports teams is a simplification of one introduced by Ken Massey in 1997. It has evolved into a part of the famous BCS (Bowl Championship Series) model for ranking college football teams and is one factor in determining which teams play in bowl games.]

I ran the following Matlab code and got the following results:

\[
A=[ 1, -1, 0, 0; -1, 0, 1, 0; 1, 0, 0, -1; 0, 0, 1, -1; ... 0, 1, 0, -1; 1, 1, 1, 1];
b = [4; 9; 6; 3; 7; 20];
r = A\b
\]

\[
r =
\]
Based on these results, T3 is ranked first, T1 second, T2 third, and T4 fourth.