AMATH 352: Problem set 4

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1 Matlab Portion

1.1 Span

Note that a set of vectors, $v_1, v_2, v_3, \ldots$, span a space $\mathbf{V}$ of dimensions n, if the matrix composed of the vectors in each column, $\mathbf{A} = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3} \ldots]$, can be reduced using row operations to a matrix containing a sub-matrix that is an identity matrix of size $n \times n$. For example, using the `rref(A)` command:

$$\texttt{rref}\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 7 & 4 & 8 \\ -1 & 2 & 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -0.6667 & 1 \\ 0 & 1 & 0.6667 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a sub-matrix of the reduced matrix, and therefore the columns of this matrix span a space of dimension 2, but do not span a space of dimension 3 or higher. (Warning! order of columns does not matter.)

Determine the dimension of the space spanned (n = ?) by each of the following sets of vectors. Save the dimension **n**, as 'A11an.dat', 'A11bn.dat', ect.

(a) $\mathbf{S_1} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$

(b) $\mathbf{S_2} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$

(c) $\mathbf{S_3} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$

(d) $\mathbf{S_4} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Summary of saved output for this section:

- `A11an.dat`,
- `A11bn.dat`,
- `A11cn.dat`,
- `A11dn.dat`

1.2 Basis

Recall that a **basis** is a set of vectors that both *span* the vector space and are *linearly independent*.

Which of the following sets of vectors form a basis of $\mathbb{R}^2$?

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 11 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(f) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

If the set forms a basis of $\mathbb{R}^2$ find the linear combination of your basis vectors that equal vector $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \in \mathbb{R}^2$. (This is also called finding the coordinates of a vector in terms of a particular basis). You can use Matlab '\' backslash.

For example, for the natural basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, when $c_1 = 3$ and $c_2 = -5$. So the coefficient vector $\vec{c} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

For each potential basis (a-f) save the coordinate vector $\vec{c}$ in 'A12acvec.dat', 'A12bcvec.dat', ect. If the vectors are not a basis, save a single value $\vec{c} =$ `NaN`.

Summary of saved output for this section:

- `A12acvec.dat`,
- `A12bcvec.dat`,
- `A12ccvec.dat`,
- `A12dcvec.dat`,
- `A12ecvec.dat`,
- `A12fcvec.dat`

1.3 Fixed Point iteration

I have provided you a matlab script called `script_for_iteration.m` and a function file `fixed_point_iteration.m`. Use this script and function to answer the following questions.

We are interested in finding the fixed point for $f(x) = x^4 + 2x^2 - x - 3 = 0$. There are multiple ways of writing $f(x) = 0$ so that you can use fixed point iteration for each $g_k(x) = x$

Evaluate different formulations of $g(x)$

(a) $g_1(x) = (3 + x - 2x^2)^{(1/4)}$

(b) $g_2(x) = (\frac{x+3-x^4}{2})^{(1/2)}$

(c) $g_3(x) = (\frac{x+3}{x^2+2})^{(1/2)}$

(d) $g_4(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1}$

For each formulation of $g_k(x)$ do the following:

- Iterate using a fixed point iteration scheme. Use as a starting guess $p_0 = 1$. Set your tolerance to `tol = 1e-6;`. Set the maximum number of iterations to `MaxIter = 1e2;`
- Save either the final fixed point value, `p`, or NaN, if the scheme does not converge to a single fixed point, as 'A13afixedpoint.dat', 'A13bfixedpoint.dat', ect.
- Using the saved iteration values of the function (`g1save`) calculate vectors of absolute, $\epsilon_a = |p(k+1) - p(k)|$, and relative, $\epsilon_r = \frac{|p(k+1)-p(k)|}{|p(k+1)|}$ errors, between each iteration. These should be row vectors. Save as 'A13aabserror.dat' and 'A13arelerror.dat'for (a-d).
- `A13afixedpoint.dat`,
- `A13bfixedpoint.dat`,
- `A13cfixedpoint.dat`,
- `A13dfixedpoint.dat`,
- `A13aabserror.dat`, `A13arelerror.dat`
- `A13babserror.dat`, `A13brelerror.dat`
- `A13cabserror.dat`, `A13crelerror.dat`
- `A13dabserror.dat`, `A13drelerror.dat`

1.4 Fixed Point iteration part II

Do the same thing as in section 1.3, but for the function $f_2(x) = x^3+4x^2-10 = 0$ **Evaluate different formulations of** $h(x)$. Confirm for yourself that for each $h_k(x) = x$ is equivalent to $f_2(x) = 0$.

(a) $h_1(x) = x - x^3 - 4x^2 + 10$

(b) $h_2(x) = \frac{1}{2}(10 - x^3)^{(1/2)}$

(c) $h_3(x) = x - (\frac{x^3+4x^2-10}{3x^2+8x})$

(d) $h_4(x) = \left(\frac{10}{4+x}\right)^{(1/2)}$

For each formulation of $h_k(x)$ do the following:

- Iterate using a fixed point iteration scheme. Use as a starting guess $p_0 = 1.5$. Set your tolerance to `tol = 1e-6;`. Set the maximum number of iterations to `MaxIter = 1e2;`
- Save either the final fixed point value, `p`, or NaN, if the scheme does not converge to a single fixed point, as 'A14afixedpoint.dat', 'A14bfixedpoint.dat', ect.
- Using the saved iteration values of the function (`g1save`) calculate vectors of absolute, $\epsilon_a = |p(k+1) - p(k)|$, and relative, $\epsilon_r = \frac{|p(k+1)-p(k)|}{|p(k+1)|}$ errors, between each iteration. These should be row vectors. Save as 'A14aabserror.dat' and 'A14arelerror.dat'for (a-d).

- `A14afixedpoint.dat`,
- `A14bfixedpoint.dat`,
- `A14cfixedpoint.dat`,
- `A14dfixedpoint.dat`,
- `A14aabserror.dat`, `A14arelerror.dat`
- `A14babserror.dat`, `A14brelerror.dat`
- `A14cabserror.dat`, `A14crelerror.dat`
- `A14dabserror.dat`, `A14drelerror.dat`

2 Written portion

2.1 Analysis of Convergence

Plot the iteration number vs the iterated value 1.3 a and b, and 1.4 a and d.
Plot the iteration number vs relative error for 1.3 a and b and 1.4 a and d.
(total of 8 plots)
Make sure you label your axis and give you plots titles.

For your write up comment on which converge and diverge. Comment on the behavior of the iterated value and the behavior of the error for convergent and divergent cases.

There will be questions on the quiz about the behavior on these plots. Consider the derivatives of your $g_k(x)$ and $h_k(x)$, and think about the convergence criteria discussed in class (Theorem 3).